
The Planar Tree Packing Theorem

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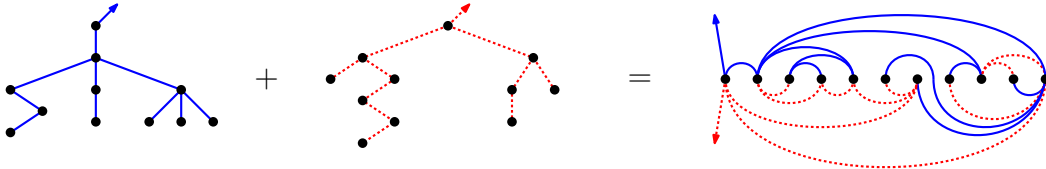
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Abstract

Packing graphs is a combinatorial problem where several given graphs are being mapped into a common host graph such that every edge is used at most once. In the planar tree packing problem we are given two trees T_1 and T_2 on n vertices and have to find a planar graph on n vertices that is the edge-disjoint union of T_1 and T_2 . A clear exception that must be made is the star which cannot be packed together with any other tree. But according to a conjecture of García et al. from 1997 this is the only exception, and all other pairs of trees admit a planar packing. Previous results addressed various special cases, such as a tree and a spider tree, a tree and a caterpillar, two trees of diameter four, two isomorphic trees, and trees of maximum degree three. Here we settle the conjecture in the affirmative and prove its general form, thus making it the planar tree packing theorem. The proof is constructive and provides a polynomial time algorithm to obtain a packing for two given nonstar trees.



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1 Introduction

The *packing problem* is to find a graph G on n vertices that contains a given collection G_1, \dots, G_k of graphs on n vertices each as edge-disjoint subgraphs. This problem has been studied in a wide variety of scenarios (see, e.g., [1, 4, 8]). Much attention has been devoted to the packing of trees (e.g., tree packing conjectures by Gyárfas [15] and by Erdős and Sós [7]). Hedetniemi [16] proved that any two nonstar trees can be packed into K_n . Teo and Yap [22] showed, extending an earlier result by Bollobás and Eldridge [2], that *any* two graphs of maximum degree at most $n - 1$ with a total of at most $2n - 2$ edges pack into K_n unless they are one of thirteen specified pairs of graphs. Maheo et al. [17] characterized triples of trees that can be packed into K_n .

In the *planar packing* problem the graph G is required to be planar. García et al. [11] conjectured in 1997 that there exists a planar packing for any two nonstar trees, that is, for any two trees with diameter greater than two. The assumption that none of the trees is a star is necessary, since a star uses all edges incident to one vertex and so there is no edge left to connect that vertex in the other tree. García et al. proved their conjecture when one of the trees is a path and when the two trees are isomorphic. Oda and Ota [20] addressed the case that one of the trees is a caterpillar or that one of the trees is a spider of diameter at most four. A *caterpillar* is a tree that becomes a path when all leaves are deleted and a *spider* is a tree with at most one vertex of degree greater than two. Frati et al. [10] gave an algorithm to construct a planar packing of any spider with any tree. Frati [9] proved the conjecture for the case that both trees have diameter at most four. Finally, Geyer et al. [13] proved the conjecture for binary trees (maximum degree three). In this paper we settle the general conjecture in the affirmative:

Theorem 1. *Every two nonstar trees of the same size admit a planar packing.*

Related work. Finding subgraphs with specific properties within a given graph or more generally determining relationships between a graph and its subgraphs is one of the most studied topics in graph theory. The *subgraph isomorphism* problem [6, 12, 24] asks to find a subgraph H in a graph G . The *graph thickness* problem [18] asks for the minimum number of planar subgraphs which the edges of a graph can be partitioned into. The *arboricity* problem [5] asks to determine the minimum number of forests which a graph can be partitioned into. Another related classical combinatorial problem is the k edge-disjoint spanning trees problem which dates back at least to Tutte [23] and Nash-Williams [19], who gave necessary and sufficient conditions for the existence of k edge-disjoint spanning trees in a graph. The interior edges of every maximal planar graph can be partitioned into three edge-disjoint trees, known as a *Schnyder wood* [21]. Gonçalves [14] proved that every planar graph can be partitioned in two edge-disjoint outerplanar graphs.

The study of relationships between a graph and its subgraphs can also be done the other way round. Instead of decomposing a graph, one can ask for a graph G that encompasses a given set of graphs G_1, \dots, G_k and satisfies some additional properties. This topic occurs with different flavors in the computational geometry and graph drawing literature. It is motivated by applications in visualization, such as the display of networks evolving over time and the simultaneous visualization of relationships involving the same entities. In the *simultaneous embedding* problem [3] the graph $G = \bigcup G_i$ is given and the goal is to draw it so that the drawing of each G_i is plane. The *simultaneous embedding without mapping* problem [3] is to find a graph G on n vertices such that: (i) G contains all G_i 's as subgraphs, and (ii) G can be drawn with straight-line edges so that the drawing of each G_i is plane.

2 Notation and Overview

A *rooted tree* is a directed tree T with exactly one vertex of outdegree zero: its root, denoted $\uparrow(T)$. Every vertex $v \neq \uparrow(T)$ has exactly one outgoing edge $(v, p_T(v))$. The target $p_T(v)$ is the *parent* of v in T , and conversely v is a *child* of $p_T(v)$. In figures we denote the root of a tree by an outgoing vertical arrow. For a vertex v of a rooted tree T , denote by $t_T(v)$ the *subtree rooted at v* , that is, the subtree of T induced by the vertices from which v can be reached on a directed path. The subscript is sometimes omitted if T is clear from the context. A *subtree of (or below) v* is a tree $t_T(c)$, for a child c of v in T . For a tree T , denote by $|T|$ the *size* (number of vertices) of T . We denote by $\deg_T(v)$ the degree (indegree plus outdegree) of v in T . For a graph G we denote by $E(G)$ the edge set of G . A *star* is a tree on n vertices that contains at least one vertex of degree $n - 1$. Such a vertex is a *center* of the star. A star on $n \neq 2$ vertices has a unique center. For a star on two vertices, both vertices act as a center. When considered as a rooted tree, there are two different rooted stars on $n \geq 3$ vertices. A star rooted at a center is called *central-star*, whereas a star rooted at a leaf that is not a center is called a *dangling star*. In particular, every star on one or two vertices is a central-star. A *nonstar* is a graph that is not a star. A *substar* of a graph is a subgraph that is a star. A *one-page book embedding* of a graph G is an embedding of G into a closed halfplane such that all vertices are placed on the bounding line. This line is called the *spine* of the book embedding.

We embed vertices equidistantly along the positive x -axis and refer to them by their x -coordinate, that is, $P = \{1, \dots, n\}$. An *interval* $[i, j]$ in P is a sequence of the form $i, i + 1, \dots, j$, for $1 \leq i \leq j \leq n$, or $i, i - 1, \dots, j$, for $1 \leq j \leq i \leq n$. Observe that we consider an interval $[i, j]$ as oriented and so we can have $i > j$. Denote the *length* of an interval $[i, j]$ by $||[i, j]|| = |i - j| + 1$. A *suffix* of an interval $[i, j]$ is an interval $[k, j]$, for some $k \in [i, j]$. To avoid notational clutter we often identify points from P with vertices embedded at them.

Overview. We construct a plane drawing of two n -vertex trees T_1 and T_2 on the point set $P = [1, n]$. We call T_1 the *blue tree*; its edges are shown as solid blue arcs in figures. The tree T_2 is called the *red tree*; its edges are shown as dotted red arcs. The algorithm first computes a preliminary one-page book embedding of T_1 onto P (the *blue embedding*) in Section 3. In the second step we recursively construct an embedding for the red tree to pair up with the blue embedding. In principle we follow a similar strategy as in the first step, but we take the constraints imposed by the blue embedding into account. During this process we may reconsider and change the blue embedding locally. For instance, we may *flip* the embedding of some subtree of T_1 on an interval $[i, j]$, that is, reflect the embedded tree at the vertical line $x = \frac{i+j}{2}$ through the midpoint of $[i, j]$. In some cases we also perform more drastic changes to the blue embedding. In particular, the blue embedding may not be a one-page book embedding in the final packing. Although neither of the two trees T_1 and T_2 we start with is a star, it is possible—in fact, unavoidable—that stars appear as subtrees during the recursion. We have to deal with stars explicitly whenever they arise, because the general recursive step works for nonstars only. We introduce the necessary concepts and techniques in Section 4 and give the actual proof in Section 5.

3 A preliminary blue embedding

We begin by defining a preliminary one-page book embedding $\pi : V_1 \rightarrow [1, n]$ for a tree $T_1 = (V_1, E_1)$ rooted at $r_1 \in V$. In every recursive step, we are given a tree T rooted at a vertex r and an interval $[i, j]$ of length $|T|$. Recall that we may have $i < j$ or $i > j$. We place r at position i and recursively embed the subtrees of r on pairwise disjoint subintervals of $[i, j] \setminus \{i\}$. The embedding is guided by two rules illustrated in Figure 1.

- The *larger-subtree-first rule* (LSFR) dictates that for any two subtrees of r , the larger of the subtrees must be embedded on an interval closer to r . Ties are broken arbitrarily.
- The *one-side rule* (1SR) dictates that for every vertex all neighbors are mapped to the same side. That is, if $N_T(v)$ denotes the set of neighbors of v in T (including its parent), then either $\pi(u) < \pi(v)$ for all $u \in N_T(v)$ or $\pi(u) > \pi(v)$ for all $u \in N_T(v)$.

These rules imply that every subtree $T \subseteq T_1$ is embedded onto an interval $[i, j] \subseteq [1, n]$ so that $\{i, j\}$ is an edge of T and either i or j is the root of T . Together with $\pi(r_1) = 1$, these rules define the embedding (up to tiebreaking). An explicit formulation of the algorithm can be found as Algorithm 1 below and an example is depicted in Figure 1c.

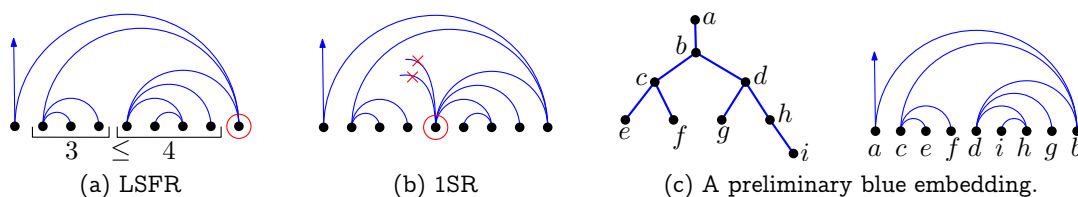


Figure 1: Illustrations for the two rules and an example embedding.

Algorithm 1: $Embed(T, I)$.

Input: A rooted tree $T = (V, E)$ and a directed interval $I \subseteq [1, n]$ with $|T| = |I|$.

Output: A map $\pi : V \rightarrow I$.

Let r be the root of T and let $[i, j] = I$.

$\pi(r) \leftarrow i$

if $|T| > 1$ **then**

 Let r_1, \dots, r_k be the children of r in T such that $|t_T(r_1)| \geq \dots \geq |t_T(r_k)|$.

$\Sigma_0 \leftarrow 0$

for $h = 1, \dots, k$ **do**

$\Sigma_h = \sum_{b=1}^h |t_T(r_b)|$.

if $i < j$ **then**

for $h = 1, \dots, k$ **do**

$Embed(t_T(r_h), [i + \Sigma_h, i + \Sigma_{h-1} + 1])$

else

for $h = 1, \dots, k$ **do**

$Embed(t_T(r_h), [i - \Sigma_h, i - \Sigma_{h-1} - 1])$

4 A red tree and a blue forest

As common with inductive proofs, we prove a stronger statement than necessary. This stronger statement does not hold unconditionally but we need to impose some restrictions on the input. The goal of this section is to derive this more general statement—formulated as Theorem 3—from which Theorem 1 follows easily.

Our algorithm receives as input a nonstar subtree R of the red tree and an interval $I = [i, j]$ of size $|R|$ along with a blue graph B embedded on I . Without loss of generality we assume $i < j$. In the initial call B is a tree, but in a general recursive call B is a *blue forest* that may consist of several components. For $k \in [i, j]$ let $B\langle k \rangle$ denote the component of B that contains k . For $[x, y] \subseteq [i, j]$ let $B[x, y]$ denote the subgraph of B induced by the vertices in $[x, y]$, and for $k \in [x, y]$ let $B[x, y]\langle k \rangle$ denote the component of $B[x, y]$ that contains k .

In general the algorithm sees only a small part of the overall picture because it has access to the vertices in I only. However, blue vertices in I may have edges to vertices outside of I and also vertices of R may have neighbors outside of I . We have to ensure that such *outside edges* are used by one tree only and can be routed without crossings. In order to control the effect of outside edges, we allow only one vertex in each component—that is, the root of R and the root of each component of B —to have neighbors outside of I . Whenever we change the blue embedding we need to maintain the relative order of these roots so as to avoid crossings among outside edges.

Conflicts. Typically $r := \uparrow(R)$ has at least one neighbor outside of I : its parent $p_{T_2}(r)$. But r may also have children in $T_2 \setminus R$. We assume that all neighbors—parent and children—of r in $T_2 \setminus R$ are already embedded outside of I when the algorithm is called for R . There are two principal obstructions for mapping r to a point $v \in I$:

- A vertex $v \in I$ is in *edge-conflict* with r , if $\{v, r'\} \in E(T_1)$ for some neighbor r' of r in $T_2 \setminus R$. Mapping r to v would make $\{v, r'\}$ an edge of both T_1 and T_2 (Figure 2a–2b). In figures we mark vertices in edge-conflict with r by a lightning symbol ℓ .
- A vertex $v \in I$ is in *degree-conflict* with r on I if $\deg_R(r) + \deg_B(v) \geq |I|$. If we map r to v , then no child of r in R can be mapped to the same vertex as a child of v in B . With only $|I| - 1$ vertices available there is not enough room for both groups (Figure 2c).

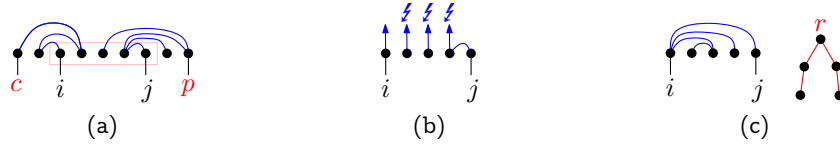


Figure 2: An interval $[i, j]$ on which a tree $R = t(r)$ is to be embedded. Two neighbors p and c of r in $T_2 \setminus R$ are already embedded (a). Then the situation on $[i, j]$ presents itself as in (b), where the three central vertices are in edge-conflict with r due to blue outside edges to p or c . In (c) the vertex i is in degree-conflict with r because $\deg_R(r) + \deg_B(i) = 2 + 3 = 5 \geq |[i, j]|$. We cannot map r to the blue vertex at i because there is not enough room for the neighbors of both in $[i, j]$.

We cannot hope to avoid conflicts entirely and we do not need to. It turns out that is sufficient to avoid a very specific type of conflict involving stars.

- An interval $[i, j]$ is in *edge-conflict* (*degree-conflict*) with $R = t(r)$ if $B^* := B\langle i \rangle$ is a central-star and the root of B^* is in edge-conflict (*degree-conflict*) with r (Figure 3).
- An interval I is in *conflict* with R if I is in edge-conflict or degree-conflict with R (or both).

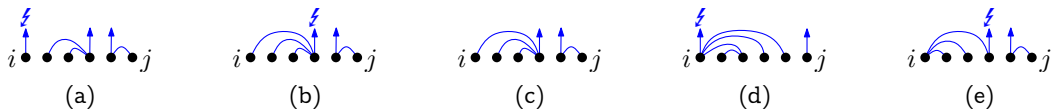


Figure 3: An interval $[i, j]$ in edge-conflict (a)–(b), and examples where $[i, j]$ is not in edge-conflict (c)–(e). In (c) the center of B^* is not in edge-conflict; it may be in degree-conflict, though, if $\deg_R(r) \geq 3$. In both (d) and (e) the tree $B\langle i \rangle$ is not a central-star.

The following lemma shows that a degree-conflict cannot be caused by a very small star.

Lemma 2. *If an interval $[i, j]$ is in degree-conflict with a nonstar subtree R of T_2 , then $B\langle i \rangle$ is a central-star on at least three vertices.*

Proof. By the definition of degree-conflict for $[i, j]$, $B^* := B\langle i \rangle$ is a central-star. Let c denote its root. Then a degree-conflict implies $\deg_R(r) + \deg_{B^*}(c) \geq |I|$. As R is not a star, we have $\deg_R(r) \leq |R| - 2 = |I| - 2$. Therefore $\deg_{B^*}(c) \geq 2$, that is, $|B^*| \geq 3$. \square

We claim that R can be packed with B onto I unless I is in conflict with R . The following theorem presents a precise formulation of this claim. Only R and the graph $B\langle i \rangle$ determine whether or not an interval $[i, j]$ is in conflict with R . Therefore we can phrase the statement without referring to an embedding of B but just regarding it as a sequence of trees. The set C represents the set of roots from B that are in edge-conflict with r .

Theorem 3. *Let R be a nonstar tree with $r = \uparrow(R)$ and let B be a nonstar forest with $|R| = |B| = n$, together with an ordering b_1, \dots, b_k of the $k \in \{1, \dots, n\}$ roots of B and a set $C \subseteq \{b_1, \dots, b_k\}$. Suppose (i) $t_B(b_1)$ is not a central-star or (ii) $b_1 \notin C$ and $\deg_R(r) + \deg_B(b_1) < n$. Then there is a plane packing π of B and R onto any interval I with $|I| = n$ such that*

- $\pi(r) \notin \pi(C)$ and
- *we can access b_1, \dots, b_k, r in this order from the outer face of π , that is, we can add a new vertex v in the outer face of π and route an edge to each of b_1, \dots, b_k, r such that the resulting multigraph is plane and the circular order of neighbors around v is b_1, \dots, b_k, r . (If $r = b_i$, for some $i \in \{1, \dots, k\}$, then two distinct edges must be routed from v to r so that the result is a non-simple plane multigraph.)*

Such a packing π we call an ordered plane packing of B and R onto I .

Theorem 3 is a strengthening of Theorem 1 and so we obtain Theorem 1 as an easy corollary.

Proof of Theorem 1 from Theorem 3. Select roots arbitrarily so that $T_1 = t(r_1)$ and $T_2 = t(r_2)$. Then use Theorem 3 with $R = T_2$, $B = T_1$, $k = 1$, $b_1 = r_1$, and $C = \emptyset$. By assumption T_1 is not a star and so (i) holds. Therefore we can apply Theorem 3 and obtain the desired plane packing of T_1 and T_2 . \square

It is not hard to see that forbidding conflicts in Theorem 3 is necessary: The example families depicted in Figure 4 do not admit an ordered plane packing.

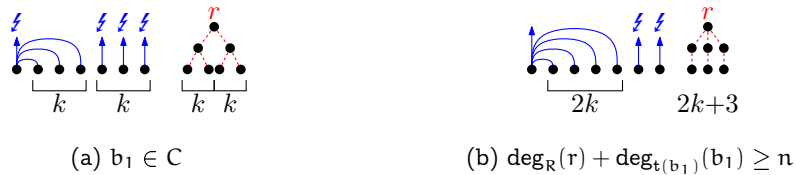


Figure 4: The statement of Theorem 3 does not hold without (i) or (ii). In the examples the trees of B are ordered from left to right so that $t(b_1)$ is a central-star. Vertices in C are labeled with ℓ .

Runtime analysis. The algorithm is parameterized with a subtree R of T_2 and an interval $I \subseteq [1, n]$, which R is to be packed onto together with an already embedded subforest of T_1 . If we represent T_1 as an adjacency matrix and the embeddings as arrays, then after an $O(n^2)$ time initialization we can test in constant time for the presence of an edge between $i, j \in I$. To represent T_2 we use an adjacency list where the children are sorted by the size of their subtrees,

which can be precomputed in $O(n \log n)$ time. Then at each step, the algorithm spends $O(|I|)$ time and makes at most two recursive calls with disjoint sub-intervals of I , which yields $O(n^2)$ time overall.

5 Embedding the red tree: fundamentals

In this section we discuss some fundamental tools for our recursive embedding algorithm to prove Theorem 3. First we formulate four invariants that hold for every recursive call of the algorithm. Next we present three tools that are specific types of embeddings to handle a “large” substar of B or R . All of these embeddings rearrange the given embedding of B to make room for the center of the star. Finally, we conclude with an outline of the algorithm.

5.1 Invariants

In the algorithm we are given a red tree $R = t(r)$, a blue forest B with roots b_1, \dots, b_k , an interval $I = [i, j] \subseteq [1, n]$ with $|I| = |R| = |B|$, and a set C that we consider to be the vertices from B in edge-conflict with r . As a first step, we embed B onto I by embedding $t(b_1), \dots, t(b_k)$ in this order from left to right, each time using the algorithm from Section 3.

Observation 4. *We may assume that R , B and $I = [i, j]$ satisfy the following invariants:*

- (I1) I is not in conflict with R . (*peace invariant*)
- (I2) Every component of B satisfies LSFR and 1SR. All edges of B are drawn in the upper halfplane (above the x -axis). All roots of B are visible from above (that is, a vertical ray going up from b_x does not intersect any edge of B). (*blue-local invariant*)
- (I3) i is not in edge-conflict with r . (*placement invariant*)

Proof. (I1) follows from the assumption (i) or (ii) in Theorem 3. (I2) is achieved by using the embedding from Section 3. If i is in conflict with r , then (I1) implies that $B\langle i \rangle$ is not a singleton (which would be a central-star). Therefore flipping $B\langle i \rangle$ establishes (I3) without affecting (I1) or (I2). \square

Theorem 3 ensures that all roots of B along with r appear on the outer face in the specified order. We cannot assume that we can draw an edge to any other vertex of B or R without crossing edges of the embedding given by Theorem 3. Therefore it is important that whenever the algorithm is called recursively,

- (I4) only the roots b_1, \dots, b_k and r have edges to the outside of I .

Assuming 1SR for B helps when splitting intervals for recursive treatment.

Observation 5. *If B satisfies (I2) and (I4) on an interval I , then both invariants also hold for $B[x, y]$ on $[x, y]$, for every subinterval $[x, y] \subseteq I$.*

In the remainder of the proof we will ensure and assume that invariants (I1)–(I4) hold for every call of the algorithm. For the initial instance of packing T_1 and T_2 , we know that (I1)–(I3) hold by Observation 4 and (I4) holds trivially because there are no vertices outside of I .

5.2 Blue-star embedding

The blue-star embedding is useful to handle the center σ of a substar B^* of B . It explicitly embeds a subtree A of R onto a part of B that includes σ . It may use some of the leaves of B^* . After taking care of σ , any unused leaf of B^* appears as a locally isolated vertex in the remaining interval of vertices.

The blue-star embedding consists of several steps: It rearranges some vertices of B , moves some edges of B below the x -axis, and introduces edges that straddle both halfplanes above and below the x -axis (Figure 5).

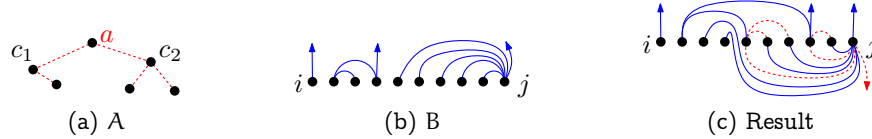


Figure 5: blue-star embedding A onto a part of B .

Suppose that A is a subtree of R with $a := \uparrow(A)$ (possibly $a = r$) and $\sigma \in [i, j]$ is the center of a star $B^* = t_B(\sigma)$. Either σ is the root of $B \langle \sigma \rangle$ or $\tau := p_B(\sigma) \in [i, j]$. Denote by B^+ the subgraph of B induced by σ and all its neighbors (parent and children). Note that either $B^+ = B^*$ or $B^+ = B^* \cup \{\tau\}$. Put $d = \deg_A(a)$ and let $\varphi = (v_1, \dots, v_d)$ be a sequence of elements from $B \setminus B^+$. Furthermore, suppose the following four conditions hold:

- (BS1) a is not in edge-conflict with σ ,
- (BS2) $|A| \leq |B^*| + \deg_A(a)$ and $|B^+| + \deg_A(a) \leq |R| - 1$,
- (BS3) at least one of $B \setminus (B^* \cup \varphi)$ or $B \setminus (B^+ \cup \varphi)$ forms an interval, and
- (BS4) if $B \setminus (B^* \cup \varphi)$ does not form an interval, then A is not a central-star, $v_1 = \tau \pm 1$ and $\{v_1, \tau\} \notin E(B)$.

Note that (BS4) is a trivial consequence of (BS3) in case $B^* = B^+$. Furthermore, (BS1) is trivially satisfied if no neighbor of a in T_2 has been embedded yet.

Let c_1, \dots, c_d denote the children of a in A such that $|t_R(c_1)| \geq \dots \geq |t_R(c_d)|$. Partition the leaves of B^* into $d + 1$ groups G_1, \dots, G_{d+1} such that $|G_k| = |t_R(c_k)| - 1$, for $k \in \{1, \dots, d\}$, and $|G_{d+1}| = |B^*| - 1 - \sum_{k=1}^d |G_k|$. We intend to embed the vertices of $t_R(c_k) \setminus \{c_k\}$ on the leaves in G_k . Note that some (possibly all) of the sets G_k may be empty. Also note that $\sum_{k=1}^d |G_k| = \sum_{k=1}^d (|t_R(c_k)| - 1) = |A| - (d + 1)$, where the $+1$ accounts for a . Therefore $|G_{d+1}| = (|B^*| - 1) - (|A| - d - 1) = |B^*| + d - |A|$ is nonnegative by (BS2) and so our assignment is well-defined.

If $B \setminus (B^* \cup \varphi)$ does not form an interval, then by (BS4) A is not a central-star and so $|G_1| \geq 1$. In this case, we move one leaf from G_1 to G_{d+1} and add τ to G_1 instead.

The *blue-star embedding of A from σ with φ* proceeds in four steps, as detailed below. The first two steps rearrange the embedding of B to make room for the embedding of A in the third step. The fourth step ensures that the remaining unused vertices appear in a form that allows to further process them.

Step 1 (Flip) We draw all edges of B^* below the spine. All edges of B not inside B^* remain above the spine (Figure 6a).

Step 2 (Mix) Leaving σ where it is, we distribute the leaves of B^* among the vertices in φ as follows: for $k \in \{1, \dots, d\}$, move the vertices of G_k so that they appear as a contiguous subsequence immediately to the right of v_k (Figure 6b). If $B \setminus (B^* \cup \varphi)$ does not form an interval, then we have τ in G_1 . As τ is not a leaf of B^* , we cannot move it around so easily. Fortunately, no relocation is necessary because by (BS4) τ appears right next to v_1 in I . Any remaining vertices in G_1 are placed between τ and v_1 .

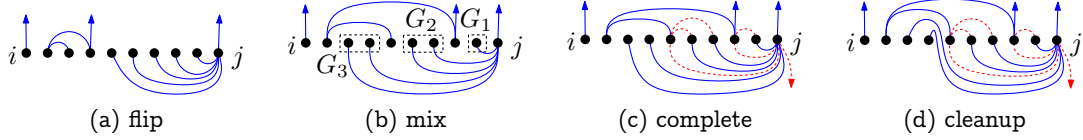


Figure 6: The example from Figure 5 in detail. We blue-star embed A from $\sigma = j$ where φ takes the vertices of $B \setminus B^+$ from right to left.

Step 3 (Complete) Embed A by first mapping a to σ , which is possible by (BS1). Next map c_i to v_i , for $i \in \{1, \dots, d\}$, drawing the edge to σ below the spine. Then embed each subtree $t_R(c_i)$ explicitly (using Algorithm 1 and drawing all edges above the spine) on the interval of $|t_R(c_i)|$ locally isolated vertices immediately to the right of c_i (Figure 6c). Note that $G_1 \cup \{v_1\}$ is locally isolated even if $B \setminus (B^* \cup \varphi)$ does not form an interval because by (BS4) we have $\{v_1, \tau\} \notin E(B)$.

It remains to describe the embedding for G_{d+1} . Before we do this, let us consider the properties that we want the embedding to fulfill. Note that the blue-star embedding—as far as described—does not use any of the invariants (I1)–(I2) other than that we start from a one-page book embedding. However, if (I1)–(I2) hold for B , then we would like to maintain these invariants also for the part $B' := B \setminus (\{\sigma\} \cup \varphi \cup \bigcup_{x=1}^d G_x)$ of B that is not yet used by R after the blue-star embedding. A necessary prerequisite is that B' forms an interval, that is, the vertices of B' appear as a contiguous subsequence of $[i, j]$. Given that we are still free to place the vertices in G_{d+1} , it is enough that the vertices in $B' \setminus G_{d+1}$ form a subinterval of $[i, j]$ that is reachable from σ (without crossing edges).

Step 4 (Cleanup) Suppose without loss of generality that σ is to the right of $B' \setminus G_{d+1}$. (If σ is to the left of $B' \setminus G_{d+1}$, replace all occurrences of “right” by “left” in the following paragraph.)

Move the vertices of G_{d+1} so that they appear as a contiguous subsequence immediately to the right of the rightmost vertex z of $B' \setminus G_{d+1}$. In order to establish that all edges are drawn above the spine, we cannot draw the edges between σ and G_{d+1} in the same way as we did for G_1, \dots, G_d above. Instead we route all edges between σ and G_{d+1} as parallel biarcs (curves that cross the spine once) that leave σ below the spine, then cross the spine just to the right of the rightmost vertex of G_{d+1} , and finally enter their destination from above (Figure 6d). As a result, for the purpose of embedding some part of R onto $[i, j - |A|]$, the vertices of G_{d+1} become isolated roots; each is connected with a single edge to the outside that is (locally) routed in the upper halfplane.

This completes the description of the blue-star embedding. Below is a formal statement summarizing the pre- and postconditions.

Proposition 6. *Let $A = t_R(a)$ be a subtree of R , let $\sigma \in [i, j]$ be the center of a star $B^* = t_B(\sigma)$, and let φ be a sequence of $\deg_A(a)$ pairwise distinct vertices from $B \setminus B^+$, where B^+ denotes the subgraph of B induced by σ and all its neighbors. If A and σ fulfill (BS1)–(BS4), then the blue-star embedding of A from σ with φ provides an ordered plane packing of A onto $[i, j] \setminus [i', j']$, for some subinterval $[i', j'] \subset [i, j]$.*

Furthermore, $\{x, \sigma\} \notin E(B)$ after the blue-star embedding, where $x = i'$, if $\sigma > j'$, and $x = j'$, if $\sigma < i'$. Put $X = B \setminus (B^ \cup \varphi)$, if $B \setminus (B^* \cup \varphi)$ is an interval, and $X = B \setminus (B^+ \cup \varphi)$,*

otherwise. Then $[i', j']$ is the union of X with some (possibly empty) sequence of isolated vertices on the side of $[i', j']$ opposite from x .

Finally, if the embedding of B on $[i, j]$ initially satisfies (I2), then after the blue-star embedding the modified embedding of B on $[i', j']$ satisfies (I2).

Proof. The packing for A is immediate by construction. Let us first argue that after the blue-star embedding an interval $[i', j'] \subset [i, j]$ remains. We distinguish two cases.

If $B \setminus (B^* \cup \varphi)$ is an interval, then the embedding uses exactly the vertices of $(B^* \cup \varphi) \setminus G_{d+1}$, and the vertices of G_{d+1} are placed so that they extend the interval $B \setminus (B^* \cup \varphi)$.

Otherwise, $B \setminus (B^* \cup \varphi)$ does not form an interval. Then the embedding uses exactly the vertices of $(B^+ \cup \varphi) \setminus G_{d+1}$ (where one vertex originally in G_1 is moved to G_{d+1}). By (BS3) we know that $B \setminus (B^+ \cup \varphi)$ forms an interval and the vertices of G_{d+1} are placed so that they extend this interval.

Next we argue that $\{x, \sigma\} \notin E(B)$. By (BS2) we have $|B \setminus B^+| = |R| - |B^+| \geq d + 1$. As φ consists of d vertices, at least one vertex in $B \setminus B^+$ is not in φ . Due to the way we run the cleanup step, it follows that the vertex of $[i', j']$ furthest from σ is in $B \setminus B^+$ (whereas the closest vertex may be in G_{d+1} , which is adjacent to σ). By construction no vertex of $B \setminus B^+$ is adjacent to σ in B . The description of $[i', j']$ holds by construction.

It remains to argue that if B satisfies (I2), then so does $B[i', j']$. The blue-star embedding does not change the order of the vertices in $B \setminus B^*$ and the vertices of G_{d+1} become isolated roots. Given the way the edges incident to G_{d+1} have been drawn, they do not affect the visibility of the roots in $B' \setminus G_{d+1}$. Therefore, (I2) holds for $B' \setminus G_{d+1}$. The validity of (I2) for the vertices in G_{d+1} follows from the discussion in Step 4 above. \square

5.3 Red-star embedding

There is a natural counterpart to the blue-star embedding that we call *red-star embedding*. It embeds a red central-star onto a blue tree.

Consider an interval $I = [i, j]$ on which we wish to embed a subtree A^* of R that is a central-star with $\alpha := \uparrow(A^*)$. Consider some $\sigma \in \{i, j\}$ such that σ is the root of $B(\sigma)$. Let $k := \deg_B(\sigma)$ and let v_1, \dots, v_k denote the children of σ in B , such that $t_B(v_1)$ is the subtree closest to σ . Choose any interval $I' \subseteq I \setminus \{\sigma\}$ such that $t_B(v_i)$ is either completely inside or completely outside I' , for every $i \in \{1, \dots, k\}$. See Figure 7a. We require that

(RS1) α is not in edge-conflict with σ and

(RS2) $\deg_{A^*}(\alpha) + \deg_{B[I' \cup \{\sigma\}]}(\sigma) \leq |I'|$.

Note that (RS1) and (RS2) are analogous to (BS1) and (BS2), but only one inequality is needed in (RS2). In the blue-star embedding, we need (BS3) and (BS4) to handle central-stars whose parent is also present in the interval under consideration. In the red-star embedding, we have no requirements on B other than (RS1) and (RS2).

Step 1 (Embed) First embed α at σ . This works by (RS1). Let $d := \deg_{A^*}(\alpha)$ and let c_1, \dots, c_d denote the children of α in A . By (RS2) the interval I' contains enough vertices not adjacent to σ in order to embed c_1, \dots, c_d . Let N be the set of the d closest non-neighbors of σ in I' . Embed c_1, \dots, c_d onto N . We next describe how to draw the red edges from c_1, \dots, c_d to α . Consider a vertex c_i and let v be the vertex of the blue forest we embedded c_i onto. Refer to Figure 7b. If $v \in t_B(v_1)$, then draw $\{c_i, \alpha\}$ as a semi-circle in the lower halfplane. If $v \in t_B(v_t)$ with $1 < t \leq k$ then draw $\{c_i, \alpha\}$ as a biarc that is in the upper halfplane near α , in the lower halfplane near c_i , and crosses the spine between v_{t-1} and $t_B(v_t)$. Finally, if $v \notin t_B(\sigma)$, then draw $\{c_i, \alpha\}$ as a biarc that is in the upper halfplane near α , in the lower halfplane near c_i , and crosses the spine right after $t_B(\sigma)$. Afterwards, the vertices of $B[I'] \setminus N$, i.e. the blue vertices that are not mapped to any c_i , are visible from below.

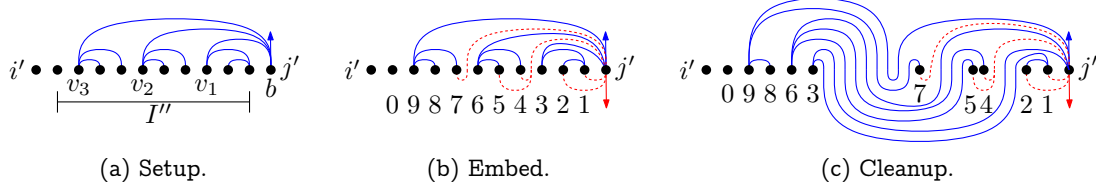


Figure 7: Using the red-star embedding to embed A^* with $\deg_{A^*}(a) = 5$.

Step 2 (Cleanup) In general, the vertices of $B[I'] \setminus N$ do not form an interval. Assume without loss of generality that σ is the rightmost vertex of I' . Let $N^+ = N \cup \{\sigma\}$. We rearrange the vertices on I' : from left to right, we first place all vertices of $B[I'] \setminus N^+$ (maintaining their relative order) and then all vertices of N^+ (maintaining their relative order). Refer to Figure 7c. In particular, σ is still at the rightmost position after this rearrangement. The edges of $B[I'] \setminus N^+$ are drawn as before, as are the edges of N^+ . We must redraw the edges that have one end vertex in N^+ and one in $B[I'] \setminus N^+$. The edges $\{v_i, \sigma\}$ are drawn as triarcs: the edge is in the upper halfplane near v_i and σ . Its first spine intersection is to the right of the rightmost vertex of $B[I'] \setminus N^+$. Its second spine intersection is such that it maintains the cyclic order of edges leaving σ (as before the rearrangement). The other edges are drawn similarly.

The pre- and postconditions of the red-star embedding are summarized by the following proposition.

Proposition 7. *Let I be an interval for which B satisfies (I2). Let $A^* = t_R(a)$ be a subtree of R that is a central-star. Consider some $\sigma \in I$ such that σ is the root of $B(\sigma)$. Let $k := \deg_B(\sigma)$ and denote the children of σ in B by v_1, \dots, v_k . Let $I' \subseteq I \setminus \{\sigma\}$ be any interval such that $t_B(v_i)$ is either completely inside or completely outside I' , for every $i \in \{1, \dots, k\}$.*

If A^ and σ and $B[I']$ fulfill (RS1) and (RS2), then the red-star embedding of A^* from σ on I' provides an ordered plane packing of A^* onto a subinterval X of I' . The set $I' \setminus X$ forms an interval that satisfies (I2) and consists of $I' \setminus I$ followed by some vertices (possibly zero) originally in $B[I']$.*

Proof. As argued above, Step 1 produces a plane packing of A^* and $B[I']$ by (RS1) and (RS2). Any remaining vertices of $B[I'] \setminus \langle \sigma \rangle$ remain visible from below. Furthermore, if a subtree of $B[I'] \setminus \langle \sigma \rangle$ is embedded onto a (directed) interval $[x, y]$ with the root at x , then Step 1 embeds children of a on a (possibly empty) suffix of $[x + 1, y]$. Since Step 2 does not change the relative position of the remaining vertices of $B[I'] \setminus \langle \sigma \rangle$ nor the relative position of the other vertices in I' , the set $I' \setminus X$ satisfies (I2) after Step 2. \square

5.4 Leaf-isolation shuffle

While we are at discussing how to deal with red stars, let us introduce another basic operation that will turn out useful in this context.

Suppose we need to embed a substar $A^* \subset R$ onto a subinterval $[a, b] \subset [i, j]$. Then we need to pair the center of A^* with an isolated vertex in $B[a, b]$. If there is no such vertex, we occasionally embed A^* onto $[a + 1, b + 1]$ after a rearrangement of B that ensures that $B[a + 1, b + 1]$ has a suitable isolated vertex. The goal of such a *leaf-isolation shuffle* is to modify B so that a leaf of $B[a, b]$ is at $a + 1$ and its parent is at a . Figure 8c shows the result of performing a left-isolation shuffle on Figure 8a with $[a, b] = [1, 9]$. The idea is then to take the parent out of the interval by embedding A^* onto $[a + 1, b + 1]$ instead and mapping the center of A^* to $a + 1$, which is locally isolated on $[a + 1, b + 1]$. The proposition below guarantees that such a leaf-isolation shuffle is always possible. Note that we do not care about the invariant (I2) in this scenario because we cannot use a recursive embedding for a star anyway. There is

one part of the invariant that we need to maintain, though, which is the visibility of the blue root from above.

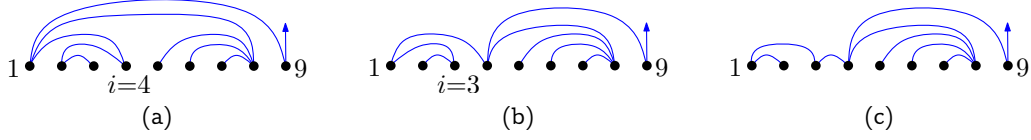


Figure 8: A leaf is shuffled into position 2, with its parent at 1.

Below is a formal statement summarizing the conditions and properties of the leaf-isolation shuffle.

Proposition 8. *Every rooted tree T on $|T| \geq 2$ vertices admits a one-page book embedding onto $[1, |T|]$ such that $q := \uparrow(T)$ is visible from above, 2 is a leaf ℓ of T , and 1 is the parent of ℓ . Moreover, if T is a central-star, then $q = 1$; otherwise, $q = |T|$.*

Proof. We use induction on $n = |T|$. Clearly the statement holds for $n = 2$. For $n \geq 3$ we start by constructing a one-page book embedding for T with a modified version of Algorithm 1 where we invert the order of subtrees, that is, we use a “smaller subtree first rule” (SSFR). By starting from q and placing it at $|T|$ we ensure that it is visible from above. As T is a tree, the embedding uses the edge $\{1, |T|\}$. If 1 is a leaf of T , then q is its parent and by SSFR T is a central-star. Therefore, flipping T yields the desired embedding. Otherwise, let $i \in \{2, \dots, |T| - 1\}$ denote the smallest (index) neighbor of 1 and obtain the desired embedding inductively for $B[1, i]$ (whose root is 1). The root of this subtree $B[1, i]$ ends up at either 1 or i , both of which are visible from above. Therefore, we can complete the embedding by routing all edges from 1 or i to the existing forest on $[i + 1, |T|]$. Figure 8 illustrates the execution of the leaf-isolation shuffle on an example. The root q is at 1 if and only if T is a central-star; otherwise, it remains at $|T|$. \square

5.5 Algorithm outline

Recall that we are given a red tree $R = t(r)$, a blue forest B with roots b_1, \dots, b_k , an interval $I = [i, j] \subseteq [1, n]$ with $|I| = |R| = |B|$, and a set $C \subset B$ of vertices in edge-conflict with r .

Let s denote a child of r that minimizes $|t_R(c)|$ among all children c of r in R . Denote $S = t_R(s)$ and $R^- = R \setminus S$. If $|R^-| \geq 2$, then R^- cannot be a central-star: if it were, then $|S| = 1$ and R would be a star. Another easy consequence of the choice of s is the following.

Lemma 9. *If $\deg_R(r) \geq 2$, then $|R^-| \geq |S| + \deg_{R^-}(r)$.*

Proof. Set $d := \deg_R(r) = \deg_{R^-}(r) + 1$ and suppose to the contrary that $|R^-| - |S| \leq \deg_{R^-}(r) - 1 = d - 2$. Adding $2|S|$ on both sides of the inequality yields $|R| \leq d + 2|S| - 2$. By the minimality of S we have $|S| \leq (|R| - 1)/d$. Solving for $|R|$ and combining with the previous inequality yields

$$d|S| + 1 \leq |R| \leq d + 2|S| - 2 \implies (d - 2)|S| \leq d - 3,$$

which is impossible, given that $|S| \geq 1$. \square

Ideally, we can recursively embed S onto $[j, j - |S| + 1]$ and R^- onto $[i, j - |S|]$ (Figure 9a). But in general the invariants may not hold for the recursive subproblems. For instance, some of the subgraphs could be stars, or if $\{i, j\} \in E(B)$, then placing r at i may put $[j, j - |S| + 1]$ in edge-conflict with S . Therefore, we explore a number of alternative strategies, depending on which—if any—of the four forests R^- , S , $B[i, j - |S|]$ and $B[j - |S| + 1, j]$ in our decomposition is a star.

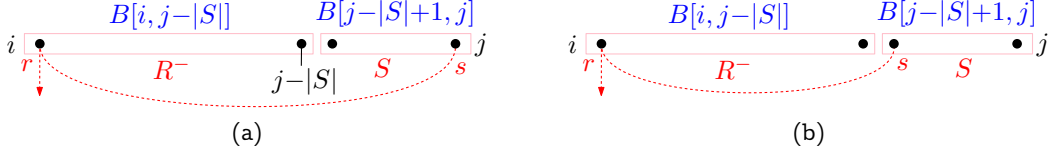


Figure 9: Our recursive strategy in an ideal world.

To complete the proof of Theorem 3 we distinguish seven cases. In each of these seven cases, we follow the notation of and assume the preconditions discussed above. First, in Section 6 we discuss the general case, where none of the four forests is a star. Then, in Section 7 and Section 8 we handle the special cases $\deg_R(r) = 1$ and $|S| = 1$, respectively. The final four sections each correspond to one of the four forests being a star. Capturing the general intuition we refer to R^- as “large” and to S as “small”, although they may have almost the same size and—in special cases, like $\deg_R(r) = 1$ — S may actually be larger than R^- .

6 Embedding the red tree: the general case

In the general case, we suppose that none of the subtrees in our current decomposition is a star.

Lemma 10. *If none of S , R^- , $B[i, j - |S|]$, and $B[j - |S| + 1, j]$ is a star, then there is an ordered plane packing of B and R onto I .*

Proof. As S is a minimum size subtree of r in R , and neither S nor R^- is a star, we know that r has at least one more subtree other than S and every subtree of r in R has size at least four. (All trees on three or less vertices are stars.) It follows that

$$\deg_{R^-}(r) \leq (|R^-| - 1)/4. \quad (1)$$

The general plan is to use one of the following two options. In both cases we first embed R^- recursively onto $[i, j - |S|]$. Then we conclude as follows.

Option 1: Embed S recursively onto $[j, j - |S| + 1]$ (Figure 9a).

Option 2: Embed S recursively onto $[j - |S| + 1, j]$ (Figure 9b).

In some cases neither of these two options works and so we have to use a different embedding.

As we embed S after R^- , the (final) mapping for s is not known when embedding R^- . However, we need to know the position of s in order to determine the conflicts for embedding R^- . Therefore, before embedding R^- we *provisionally* embed s at $\alpha := \uparrow(B[j - |S| + 1, j])(j)$ (Option 1) or $\alpha := \uparrow(B[j - |S| + 1, j])(j - |S| + 1)$ (Option 2). That is, for the recursive embedding of R^- we pretend that some neighbor of r is embedded at α . In this way we ensure that S is not in edge-conflict with the interval in its recursive embedding. The final placement for s is then determined by the recursive embedding of S , knowing the definite position of its parent r .

For the recursive embeddings to work, we need to show that the invariants (I1), (I2) and (I4) hold ((I3) then follows as in Observation 4). For (I2) and (I4) this is obvious by construction and Observation 5, as long as we do not change the embedding of B . As we do not change the embedding in Option 1 and 2, it remains to ensure (I1). So suppose that for both options, (I1) does not hold for at least one of the two recursive embeddings. There are two possible obstructions for (I1): edge-conflicts and degree-conflicts. We discuss both types of conflicts, starting with edge-conflicts.

Case 1 $[i, j - |S|]$ is not in degree-conflict with R^- and $[j, j - |S| + 1]$ is not in degree-conflict with S . Then Option 1 works, unless $[i, j - |S|]$ is in edge-conflict with R^- . Recall that $[j, j - |S| + 1]$ is not in edge-conflict with S after embedding R^- onto $[i, j - |S|]$ due to the provisional placement of s .

We claim that an edge-conflict between R^- and $[i, j - |S|]$ implies $\{i, j\} \in E(B)$. To prove this claim, suppose that $[i, j - |S|]$ is in edge-conflict with R^- . Then $B[i, j - |S|](i)$ is a central-star whose root c is in edge-conflict with r . If $c = i$, then by (I3) there was no such conflict initially (for R and $[i, j]$). So, as claimed, the conflict can only come from a blue edge to s (provisionally placed) at j . Otherwise, $c > i$ and by 1SR there is no edge in B from c to any point in $[c + 1, j]$. It follows that $B[i, j - |S|](i) = B(i)$. The conflict between c and r does not come from the edge to s but from an edge to a vertex outside of $[i, j]$. This contradicts (I1) for R and $[i, j]$, which proves the claim.

The presence of the edge $\{i, j\}$ implies that B is a tree and by (I4) only (the root) i or j may have edges out of $[i, j]$. Consider Option 2, which embeds S onto $[j - |S| + 1, j]$, provisionally placing s at $\uparrow(B[j - |S| + 1, j](j - |S| + 1))$. There are two possible obstructions: an edge-conflict for R^- or a degree-conflict for S . In both cases we face a central-star $B^* = B[j - |S| + 1, b]$ with center $b \in [j - |S| + 1, j - 1]$. Due to 1SR and $\{i, j\} \in E(B)$, we know that $b = \uparrow(B[j - |S| + 1, j](j - |S| + 1))$. We distinguish three cases.

Case 1.1 $\{i, b\} \in E(B)$. Then we consider a third option: provisionally place s at j , embed R^- recursively onto $[j - |S|, i]$ and then S onto $[j, j - |S| + 1]$ (Figure 10a). The edge $\{i, b\}$ of B prevents any edge-conflict between $[j - |S|, i]$ and R^- (and, as before, for S). Given that we assume in Case 1 that $[j, j - |S| + 1]$ is not in degree-conflict with S , we are left with $[j - |S|, i]$ being in degree-conflict with R^- as a last possible obstruction.

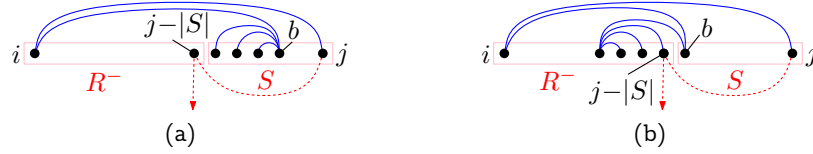


Figure 10: A third embedding when the first two options fail.

Then the tree $B[i, j - |S|](j - |S|)$ is a central-star A^* with root a such that

$$\deg_{A^*}(a) + \deg_{R^-}(r) \geq |R^-|. \quad (2)$$

Combining Lemma 9 with (2) we get $|A^*| = \deg_{A^*}(a) + 1 \geq |S| + 1 \geq 5$. Note that A^* can be huge, but we know that it does not include i (because $B[i, j - |S|]$ is not a star). We also know that $a \neq j - |S|$: If $a = j - |S|$, then by 1SR we have $p_B(a) \in [i, j - |S| - 1]$, in contradiction to $a = \uparrow(B[i, j - |S|](j - |S|))$. Therefore $a = j - |S| - |A^*| + 1$ and by 1SR its parent is to the right. Due to $\{i, b\} \in E(B)$ and since $B[j - |S| + 1, b]$ is a tree rooted at b , we have $p_B(a) = b$. As A^* is a subtree of b in B on at least five vertices, by LSFR b cannot have a leaf at $b - 1$. Therefore, the star $B[j - |S| + 1, j](j - |S| + 1)$ consists of a single vertex only, that is, $b = j - |S| + 1$ (Figure 10b). We consider two subcases. In both the packing is eventually completed by recursively embedding S onto $[j, j - |S| + 1]$.

Case 1.1.1 $\{x, b\} \in E(B)$, for some $x \in [i + 1, a - 1]$ (Figure 11a). Select x to be maximal with this property. Then we exchange the order of the two subtrees $t(x)$ and A^* of b (Figure 11b). This may violate LSFR for B at b , but (I2) holds for both $B[i, j - |S|]$ and $B[j - |S| + 1, j]$. Clearly there is still no edge-conflict for $[j - |S|, i]$ with R^- after this change. We claim that there is no degree-conflict anymore, either.

To prove the claim, note that by LSFR at b we have $|t(x)| \leq |A^*|$. As the size of both subtrees combined is at most $|R^-|$, we have $|t(x)| \leq |R^-|/2$. Then, using (1), $|t(x)| - 1 + \deg_{R^-}(r) <$

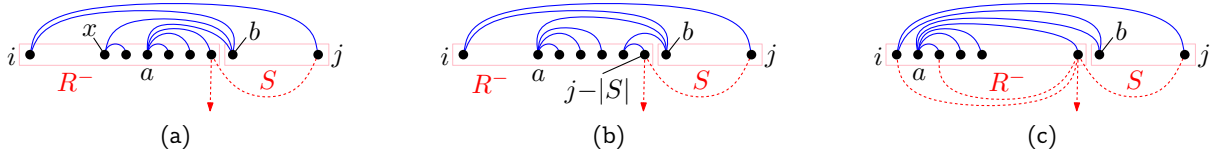


Figure 11: Swapping two subtrees of b in Case 1.1.1 and an explicit embedding for Case 1.1.2.

$|R^-|/2 + \deg_{R^-}(r) < 3|R^-|/4 < |R^-|$. Therefore after the exchange $[j - |S|, i]$ is not in degree-conflict with R^- , which proves the claim and concludes this case.

Case 1.1.2 i and $a = j - |S| - |A^*| + 1$ are the only neighbors of b in B . We claim that in this case A^* extends all the way up to $i + 1$, that is, $A^* = B[i + 1, j - |S|]$. To prove this claim, suppose to the contrary that $a \geq i + 2$. Then there is another subtree of i to the left of a and, in particular, $\{i, a - 1\} \in E(B)$. By LSFR this closer subtree is at least as large as A^* . Using (1) and (2) we get $||i + 1, a - 1|| + |A^*| \geq 2|A^*| > 2(|R^-| - \deg_{R^-}(r)) > 3|R^-|/2 > |R^-|$, in contradiction to $||i + 1, a - 1|| + |A^*| < |R^-|$. Therefore $a = i + 1$, as claimed (Figure 11c).

The vertex a has high degree in B but it is not adjacent to i . Therefore, we can embed R^- as follows: put r at $j - |S|$ and embed an arbitrary subtree Y of r onto $[i, i + |Y| - 1]$ recursively or, if it is a star, explicitly, using the locally isolated vertex at i for the center (and $i + |Y| - 1$ for the root in case of a dangling star). As i is isolated on $[i, i + |Y| - 1]$ there is no conflict between $[i, i + |Y| - 1]$ and Y . As $|Y| \geq |S| \geq 4$, the remaining graph $B[i + |Y|, j - |S| - 1]$ consists of isolated vertices only, on which we can explicitly embed any remaining subtrees of r using the algorithm from Section 3.

Case 1.2 $\{i, b\} \notin E(B)$ and $b = p_B(j - |S|)$. Then $j - |S|$ is a locally isolated vertex in $B[i, j - |S|]$, whose only neighbor in B is at $b \notin B[j - |S| + 1, j]$. Therefore, we can provisionally place s at j so that $[j - |S|, i]$ is not in conflict with R^- . By the assumption of Case 1 $[j, j - |S| + 1]$ is not in degree-conflict with S . Therefore, we obtain the claimed packing by first embedding R^- onto $[j - |S|, i]$ recursively and then S onto $[j, j - |S| + 1]$.

Case 1.3 $\{i, b\} \notin E(B)$ and $b \neq p_B(j - |S|)$. As $\{i, b\} \notin E(B)$ and s is provisionally placed at b , the interval $[i, j - |S|]$ is not in edge-conflict with R^- . Thus, Option 2 (Figure 9b) succeeds unless $[j - |S| + 1, j]$ is in degree-conflict with S . Hence suppose

$$\deg_S(s) + \deg_{B^*}(b) \geq |S|. \quad (3)$$

By Lemma 2 we have $|B^*| \geq 3$. As $b \neq p_B(j - |S|)$, by LSFR b has exactly one neighbor in B outside of B^* : its parent $p_B(b) \in [i + 1, j - |S|]$ (Figure 12). Let $B^+ = B^* \cup \{p_B(b)\}$. We blue-star embed S starting from b with $\varphi = (v_1, \dots, v_d) = (j, \dots)$ so that φ takes the vertices of $I \setminus B^+$ from right to left. Let us argue that the conditions for the blue-star embedding hold.

(BS1) holds due to $\{i, b\} \notin E(B)$ and $i = \uparrow(B[i, j - |S|](i))$. For the first inequality of (BS2) we have to show $|S| \leq |B^*| + \deg_S(s)$, which is immediate from (3). For the second inequality of (BS2) we have to show $|B^+| + \deg_S(s) \leq |I| - 1$. This follows from $|B^*| + 1 + \deg_S(s) \leq (|S| - 1) + 1 + (|S| - 1) \leq |S| + (|R^-| - 1) - 1 = |I| - 2$. Regarding (BS3) note that in φ we take the vertices of $I \setminus B^+$ from right to left. If φ reaches beyond $p_B(b)$, then $B \setminus (B^+ \cup \varphi)$ forms an interval (Figure 12b); otherwise, $B \setminus (B^* \cup \varphi)$ forms an interval (Figure 12e). Conversely, if $B \setminus (B^* \cup \varphi)$ does not form an interval, then φ reaches beyond $p_B(b)$. In particular, in that case φ includes $p_B(b) - 1$ and we may simply move $p_B(b) - 1$ to the front of φ , establishing the second condition in (BS4). Regarding the remaining two conditions it suffices to note that S is not a star by assumption and that $p_B(b) - 1$ is not a neighbor of b in B because $p_B(b)$ is the only neighbor of b outside of B^* .

Therefore, we can blue-star embed S as claimed. By construction and Proposition 6 that leaves us with an interval $[i', j']$, where $i = i'$. This “new” interval is obtained from the interval

$[i, j - |S|]$ before the blue-star embedding by replacing some suffix of vertices by a corresponding number of locally isolated vertices. In particular, $B[i', j']\langle i' \rangle$ is a subtree of $B[i, j - |S|]\langle i \rangle$ and $i' = \uparrow(B[i', j']\langle i' \rangle)$.

We complete the packing by recursively embedding R^- onto $[i', j']$. This interval is not in edge-conflict with R^- by (I3), $\{i, b\} \notin E(B)$ and $i' = \uparrow(B[i', j']\langle i' \rangle)$. We claim that it is not in degree-conflict with R^- , either. Suppose towards a contradiction that $[i', j']$ is in degree-conflict with R^- . Then $B[i', j']\langle i' \rangle$ is a central-star and so by LSFR also $B[i, j - |S|]\langle i \rangle$ is a central-star on at least this many vertices before the blue-star embedding. This contradicts the assumption of Case 1 that $[i, j - |S|]$ is not in degree-conflict with R^- . Therefore, $[i', j']$ is not in degree-conflict with R^- and we can complete the packing as described. This completes the proof for Case 1.

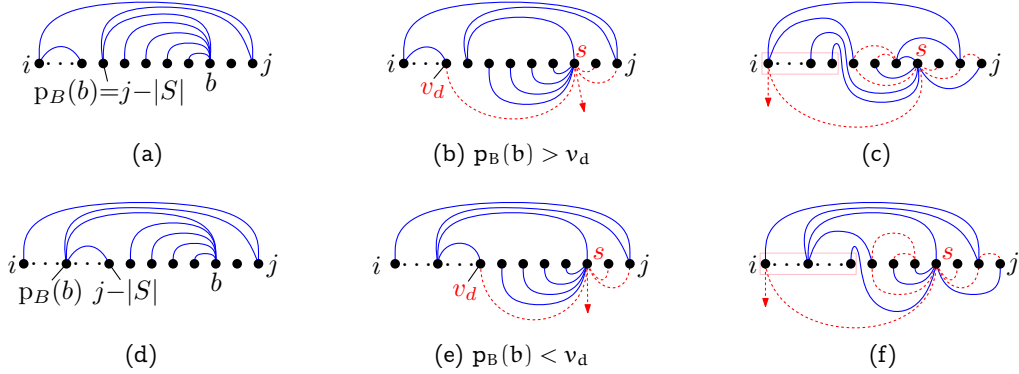


Figure 12: Explicit embedding of S in Case 1.3. The edge $\{p_B(b), j\}$ need not be present in B .

Case 2 $[i, j - |S|]$ is in degree-conflict with R^- . Then $B[i, j - |S|]\langle i \rangle$ is a central-star $B[i, x]$

$$\text{with } \deg_{R^-}(r) + (x - i) \geq |R^-| \quad (4)$$

and $|B[i, x]| = x - i + 1 \geq 3$ by Lemma 2. We distinguish two cases.

Case 2.1 $B\langle i \rangle = B[i, x]$. Then $B\langle i \rangle \neq B\langle j \rangle$. If necessary, flip $B\langle i \rangle$ to put its center at i . If $B\langle j \rangle$ is a central-star on ≥ 3 vertices, then—if necessary—flip $B\langle j \rangle$ to put its root at j . We use a blue-star embedding for R^- starting from $\sigma = i$ with $\varphi = (x + 1, \dots)$. As φ consists of $d := \deg_{R^-}(r)$ vertices, we have $[i, j] \setminus (B[i, x] \cup \varphi) = [x + d + 1, j]$. If $B[x + d + 1, j]\langle j \rangle$ is a central-star on ≥ 3 vertices, then use $\varphi = (j, x + 1, \dots)$ instead (and note that $\uparrow(B[x + d + 1, j]\langle j \rangle) = j$).

In the notation of the blue-star embedding we have $B^* = B^+ = B[i, x]$. We need to show that the conditions for this embedding hold. (BS1) holds by (I1) (for embedding R onto $[i, j]$). For (BS2) we have to show $|R^-| \leq |B^*| + \deg_{R^-}(r) \leq |R| - 1$. The first inequality holds by (4) and $|B^*| \geq x - i$. The second inequality holds due to (I1) (for embedding R onto $[i, j]$), which implies $\deg_R(r) + (x - i) \leq |R| - 1$. As $|B[i, x]| = x - i + 1$ and $\deg_R(r) = \deg_{R^-}(r) + 1$, (BS2) follows. (BS3) is obvious by the choice of φ and (BS4) is trivial for $B^* = B^+$ due to (BS3). That leaves us with an interval $[i', j']$, where $j' \in \{j, j - 1\}$. We claim that $[j', i']$ is not in conflict with S .

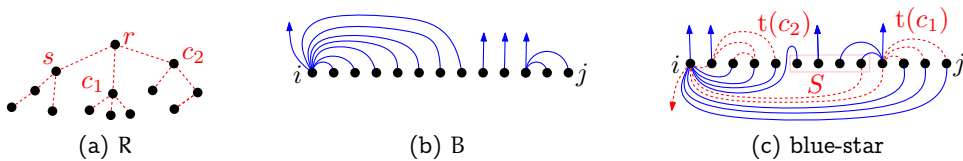


Figure 13: Handling a degree-conflict for R^- in Case 2.1.

To prove the claim we consider two cases. If $j' = j - 1$, then initially $B[x + d + 1, j]\langle j \rangle$ was a central-star on ≥ 3 vertices rooted at j . By the choice of φ a leaf of this star is at j' whose only neighbor in B is at $j \neq i = r$. Therefore $[j', i']$ is not in edge-conflict with S . As $B[j', i']\langle j' \rangle$ is an isolated vertex, by Lemma 2 there is no degree-conflict between $[j', i']$ and S , either, which proves the claim.

Otherwise, $j' = j$ and $B[j', i']\langle j' \rangle = B[x + d + 1, j]\langle j \rangle$ is not a central-star on ≥ 3 vertices. Therefore by Lemma 2 there is no degree-conflict between $[j', i']$ and S . In order to show that there is no edge-conflict, either, it is enough to show that $\uparrow(B[j', i']\langle j' \rangle)$ is not adjacent to $i = r$ in B . If $\uparrow(B[j', i']\langle j' \rangle) \neq j'$ this follows from 1SR. Otherwise $\uparrow(B[j', i']\langle j' \rangle) = j' = j$, and $\{i, j\} \notin E(B)$ because $B\langle i \rangle$ is a star but B is not. Therefore the claim holds and we can complete the packing by recursively embedding S onto $[j', i']$.

Case 2.2 $B\langle i \rangle \neq B[i, x]$. By 1SR this means that $i = p_B(x)$ and i has at least one more neighbor in $[i, j] \setminus B[i, x]$. Since by assumption $B[i, j - |S|]$ is not a star, we have $x \leq j - |S| - 1$. Since $B[i, x]$ is a central-star and $x \leq j - |S| - 1$, by LSFR for i the only neighbor of i in B outside of $B[i, x]$ is its parent $p_B(i) \in [j - |S| + 1, j]$. We claim that such a configuration is impossible. To prove the claim, note that $p_B(i)$ has at least two children in $B[i, j - |S|]$ because $x \leq j - |S| - 1$ and $p_B(i) \geq j - |S| + 1$. By LSFR, the corresponding subtrees have size at least $|B[i, x]| = x - i + 1$, and so $|R| \geq |B[i, p_B(i)]| \geq 2|B[i, x]| + 1 \geq 2(|R^-| - \deg_{R^-}(r) + 1) + 1$, where the last inequality uses (4). Rewriting and using (1) yields

$$|R^-| \leq \frac{|R| - 1}{2} + \deg_{R^-}(r) - 1 < \frac{|R| - 1}{2} + \frac{|R^-|}{4}.$$

It follows that $|R^-| < \frac{2}{3}(|R| - 1)$ and hence that $|S| > \frac{1}{3}(|R| - 1)$. Since S is a smallest subtree of r in R , this means that r is binary in R and thus unary in R^- . This, finally, contradicts the degree-conflict for $[i, j - |S|]$ with R^- because $x < j - |S|$ and hence $\deg_{R^-}(r) + \deg_{B[i, x]}(i) = 1 + (x - i) < 1 + (j - |S|) - i = |R^-|$.

Case 3 $[j, j - |S| + 1]$ is in degree-conflict with S and $[i, j - |S|]$ is not in degree-conflict with R^- . Then $B[j - |S| + 1, j]\langle j \rangle$ is a central-star $Z = t_B(z)$ with $|Z| \geq 3$ by Lemma 2 and

$$\deg_S(s) + \deg_Z(z) \geq |S|. \quad (5)$$

Case 3.1 $\{i, j\} \notin E(B)$. Then we claim that we may assume $z = j$ and $Z = B\langle j \rangle$.

Let us prove this claim. If $z = j - |Z| + 1$, then by 1SR it does not have any neighbor in $B \setminus Z$. Flipping $Z = B\langle j \rangle$ establishes the claim. Otherwise, $z = j$. Suppose that z has a neighbor $y \in B \setminus Z$. As z is the root of $Z = B[j - |S| + 1, j]\langle j \rangle$, it does not have a neighbor in $[j - |S| + 1, j - |Z|]$ and therefore $y \in [i + 1, j - |S|]$. By LSFR and because $B[j - |S| + 1, j]$ is not a star, $y = p_B(z)$. In particular, since $|Z| \geq 3$, LSFR for y implies $\{y, y + 1\} \notin E(B)$. It follows that after flipping $B\langle j \rangle$ the resulting subtree $B[j - |S| + 1, j]\langle j \rangle$ is not a central-star anymore and so there is no conflict for embedding S onto $[j, j - |S| + 1]$ anymore. Therefore we can proceed as above in Case 1 (the conflict situation for R^- did not change because $B\langle i \rangle$ remains unchanged). Hence we may suppose that there is no such neighbor y of z , which establishes the claim.

We blue-star embed S starting from $\sigma = j = z$ with $\varphi = (j - |Z|, j - |Z| - 1, \dots)$. In the terminology of the blue-star embedding we have $B^* = B^+ = Z$. Let us argue that the conditions for the embedding hold. (BS1) is trivial because no neighbor of s is embedded yet. For (BS2) we have to show $|S| \leq |Z| + \deg_S(s) \leq |R| - 1$. The first inequality holds by (5) and the second by $|S| \leq (|R| - 1) / \deg_R(r) \leq (|R| - 1) / 2$, which implies $|Z| + \deg_S(s) \leq 2(|S| - 1) \leq |R| - 3$. (BS3) is obvious by the choice of φ and given $B^* = B^+$, (BS4) is trivial. That leaves us with an interval $[i', j']$, where $i' = i$.

The plan is to recursively embed R^- onto $[i, j']$. This works fine, unless $[i, j']$ and R^- are in conflict. So suppose that they are in conflict. Then there is a central-star $Y = B[i, j']\langle i \rangle$.

Considering how φ consumes the vertices in I from right to left, Y appears as a part of some component of B , that is, $Y = B[i, y]$, for some $y \in [i, j - |S|]$.

We claim that $\uparrow(Y) = i$. To prove the claim, suppose to the contrary that $\uparrow Y = y = p_B(i)$. Then by 1SR Y is a component of B . Thus, a degree-conflict contradicts the assumption of Case 3 that $[i, j - |S|]$ is not in degree-conflict with R^- , and an edge-conflict contradicts (I1) for embedding R onto $[i, j]$ together with the fact that by 1SR y is not adjacent to any vertex outside of Y in B —in particular not to j , where s was placed. This proves the claim and, furthermore, that $p_B(i) \in [y + 1, j - |S|]$ and $p_B(i)$ appears in φ .

By (I3) for embedding R onto $[i, j]$ and $\{i, j\} \notin E(B)$ we know that $[i, j']$ and R^- are not in edge-conflict and so they are in degree-conflict. In particular, $\deg_Y(i) + \deg_{R^-}(r) \geq |R^-|$.

Undo the blue-star embedding. We claim $\{i, y + 1\} \in E(B)$. To prove the claim, suppose to the contrary that $\{i, y + 1\} \notin E(B)$. Then $\{y + 1, p_B(i)\} \in E(B)$ because in B the vertex $y + 1$ lies below the edge $\{i, p_B(i)\}$. By LSFR the subtree of $p_B(i)$ rooted at $y + 1$ is at least as large as Y . Therefore,

$$|t_B(p_B(i))| \geq 2|Y| + 1 = 2\deg_Y(i) + 3 \geq 2(|R^-| - \deg_{R^-}(r)) \geq \frac{3}{2}|R^-|,$$

where the last inequality uses (1). This is in contradiction to $p_B(i) \leq j - |S|$, which implies $|t_B(p_B(i))| \leq |R^-|$. Therefore, the claim holds and $\{i, y + 1\} \in E(B)$.

Flip $B[i, y + 1]$ and perform the blue-star embedding again. Although 1SR may be violated at $y + 1$, this is of no consequence for the blue-star embedding. As $Y = B[i, j'] \setminus \{i\} = B[i, y]$, we know that $y + 1$ appears in φ and so the offending vertex is not part of $[i, j']$ after the blue-star embedding. Furthermore, in this way we also get rid of the high-degree vertex of B that was at i initially so that the vertices in $[i, y] \subset [i, j']$ are isolated. In particular, i is isolated in $[i, j']$ and its only neighbor in B is at $y + 1 \neq j$. Therefore, $[i, j']$ and R^- are not in conflict, unless $y + 1 = p_B(i)$ initially and $p_B(i)$ is in edge-conflict with r .

In other words, it remains to consider the case $B(i) = B[i, y + 1] = t_B(y + 1)$ is a dangling star whose root i (at $y + 1$ before flipping) is in edge-conflict with r (Figure 14a). Then $[i, j']$ is an independent set in B that consists of leaves of the two stars Y and Z plus the isolated vertex at i . Yet we cannot simply embed R^- using the algorithm from Section 3 because i is and j' may be in edge-conflict with r . Given that $|Y| \geq 3$ and φ gets to $y + 1$ only, at least two leaves of Y remain in $[i, j']$ and so, in particular, $i + 1$ is not in conflict with r . We explicitly embed R^- as follows (Figure 14b): place r at $i + 1$ and a child c of r in R^- at i . Then collect $|t_R(c)|$ leaves from Z and/or Y and put them right in between i and $i + 1$. First—from left to right—the leaves of Z whose blue edges leave them upwards to bend down and cross the spine immediately to the right of the vertices of the red subtree rooted at $y + 1$ (the leftmost subtree of S) and then reach z from below. Next come the leaves of Y whose blue edges to $y + 1$ are drawn as arcs in the upper halfplane. In order to make room for those leaves, the blue edge $\{i, y + 1\}$ is re-routed to leave i downwards to bend up and cross the spine immediately to the left of $i + 1$ in order to reach $y + 1$ from above. Using the algorithm from Section 3 we can now embed $t_R(c)$ onto these leaves and any remaining subtrees of r can be embedded explicitly on the vertices $i + 2, \dots$ (ignoring the change of numbering caused by the just discussed repositioning of leaves).

Case 3.2 $\{i, j\} \in E(B)$. Then $z = j$ because $j - |Z| + 1$ is enclosed by $\{i, j\}$ and therefore cannot be the root of Z . Moreover, $\uparrow(B) = i$ by LSFR and since B is not a star. By LSFR j does not have any child in $B \setminus Z$ and as $B[j - |S| + 1, j]$ is not a star, $Z \subseteq B[j - |S| + 2, j]$. In particular, j is not adjacent to any vertex in $B[i + 1, j - |S| + 1]$. We provisionally place s at any vertex in $[i + 1, j - |R^-|]$, say, at $j - |R^-|$. Then $[j, j - |R^-| + 1]$ is not in edge-conflict with R^- . We claim that it is not in degree-conflict, either. As Z is a star on $|Z| \leq |S| - 1$ vertices, by Lemma 9 we have $\deg_{R^-}(r) + \deg_Z(z) \leq \deg_{R^-}(r) + |S| - 2 \leq |R^-| - 2$ and the claim follows. We recursively embed R^- onto $[j, j - |R^-| + 1]$, treating all local roots of B other than j as in conflict with r . It remains to recursively embed S onto $[j - |R^-|, i]$.

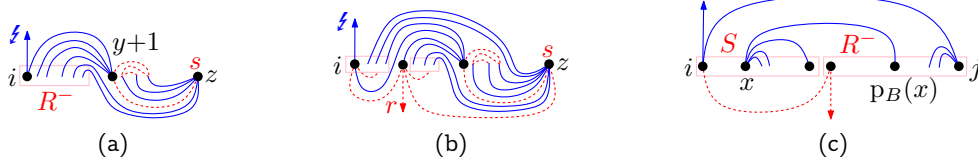


Figure 14: (a)–(b): Relocating some leaves of the stars $Y = t_B(y+1)$ and $Z = t_B(z)$ in Case 3.1. One subtree of $t_R(c)$ of R^- is embedded at i and the leaves to the right of i ; all other subtrees of R^- are embedded to the right of r . Both from $y+1$ and from z we can route as many blue edges as desired to either of these “pockets”. (c): Evading a degree-conflict for S in Case 3.2.1.

Suppose towards a contradiction that $[j - |R^-|, i]$ is in conflict with S . Then there is a central-star $X = B[x, j - |R^-|] = B[i, j - |R^-|] \langle j - |R^-| \rangle$. Due to $\{i, j\} \in E(B)$ and 1SR we have $\uparrow(X) = x$ and $p_B(x) > j - |R^-|$. Together with LSFR for j it follows that $p_B(x) \in [j - |R^-| + 1, j - |Z|]$. Due to the conflict setting for embedding R^- , i is the only vertex in $[j - |R^-|, i]$ that may be in edge-conflict with s . As X is a central-star and $\uparrow(B) = i$, we cannot have $x = i$ because then B would be a star. It follows that $x > i$ and so $[j - |R^-|, i]$ is not in edge-conflict with S . Therefore $[j - |R^-|, i]$ and S are in degree-conflict. Then $|X| \geq 3$ by Lemma 2 and

$$\deg_S(s) + \deg_X(x) \geq |S|. \quad (6)$$

Depending on $p_B(x)$ we consider two final subcases.

Case 3.2.1 $p_B(x) \in [j - |R^-| + 2, j - |Z|]$ (Figure 14c). Then the edge $\{x, p_B(x)\}$ encloses $j - |R^-| + 1$ so that, in particular, $\{i, j - |R^-| + 1\} \notin E(B)$. We provisionally place s at $i = \uparrow(B[i, j - |R^-|] \langle i \rangle)$ and claim that $[j - |R^-| + 1, j]$ and R^- are not in conflict.

To prove the claim, consider $W^* := B[j - |R^-| + 1, j] \langle j - |R^-| + 1 \rangle$ and suppose it is a central-star. (If it is not, then we are done.) If $\uparrow(W^*) > j - |R^-| + 1$, then by 1SR and $\{x, p_B(x)\} \in E(B)$ we have $p_B(\uparrow(W^*)) = x$, in contradiction to LSFR for x . Therefore $\uparrow(W^*) = j - |R^-| + 1$. In order for $j - |R^-| + 1$ to be the local root for W^* in the presence of $\{x, p_B(x)\} \in E(B)$, it follows that $p_B(j - |R^-| + 1) = x$ and so by 1SR $|W^*| = 1$. Therefore by Lemma 2 there is no degree-conflict between $[j - |R^-| + 1, j]$ and R^- . As $\{x, p_B(x)\} \in E(B)$ prevents any connection in B from $j - |R^-| + 1$ to i and to vertices outside of $[i, j]$, there is no edge-conflict between $[j - |R^-| + 1, j]$ and R^- , either. This proves the claim. Recursively embed R^- onto $[j - |R^-| + 1, j]$. Recall that $\uparrow(B) = i$. There is no conflict for embedding S onto $[i, j - |R^-|]$ since $\{i, r\} \notin E(B)$ and $B[i, j - |R^-|] \langle i \rangle$ is not a central-star of size at least 2 by LSFR at i . Finish the packing by recursively embedding S onto $[i, j - |R^-|]$.

Case 3.2.2 $p_B(x) = j - |R^-| + 1$. Then by 1SR $p_B(x)$ is the only neighbor of x outside of X in B . We provisionally place r at j and employ a blue-star embedding for S , starting from $\sigma = x$ with $\varphi = (i, \dots)$, that is, φ takes vertices from left to right, skipping over $[x, p_B(x)]$. Let us argue that the conditions for the blue-star embedding hold.

In the terminology of the blue-star embedding we have $B^* = X$ and $B^+ = X \cup \{p_B(x)\}$. (BS1) holds because $\{x, j\} \notin E(B)$. For the first inequality of (BS2) we have to show $|S| \leq |X| + \deg_S(s)$, which is immediate from (6). For the second inequality of (BS2) we have to show $|X| + 1 + \deg_S(s) \leq |I| - 1$. This follows from $|X| + 1 + \deg_S(s) \leq |S| + (|S| - 1) \leq |R^-| + |S| - 1 = |I| - 1$. Regarding (BS3) note that in φ we take the vertices of $B \setminus B^+$ from left to right. As there are not enough vertices in $[i, x - 1]$ to embed the neighbors of s (which causes the degree-conflict), φ reaches beyond $p_B(x)$ and so $B \setminus (B^+ \cup \varphi)$ forms an interval. In particular, φ includes $p_B(x) + 1$ and we may simply move $p_B(x) + 1$ to the front of φ , establishing the second condition in (BS4). Regarding the remaining two conditions in (BS4) note that S is not a star by assumption and that $p_B(x) + 1$ is not a neighbor of x in B because $p_B(x)$ is the only neighbor of x outside of X .

Therefore, we can blue-star embed S as claimed, which leaves us with an interval $[i', j']$, where $j = j'$. As $\{x, j\} \notin E(B)$ and j is not the local root of B (i is), there is no edge-conflict between $[j', i']$ and R^- . As there is no degree-conflict between $[j, j - |R^-| + 1]$ and R^- and the number of neighbors of j in $B[i', j']$ can only decrease compared to $B[j - |R^-| + 1, j]$ (if they appear in φ), there is no degree-conflict between $[j', i']$ and R^- , either. Therefore, we can complete the packing by embedding R^- onto $[j', i']$ recursively. \square

7 Embedding the red tree: a unary root

In this section we handle all cases where the root r of R is unary.

Proposition 11. *If $\deg_R(r) = 1$ and S is a star, then there is an ordered plane packing of B and R onto I .*

Proof. Since $\deg_R(r) = 1$ and R is not a star by assumption, S must be a dangling star. Thus, we know exactly what R looks like: it is rooted at r , which has a single child s , which has a single child q , which finally has zero or more leaf children.

Case 1 $\{i, j\} \notin E(B)$. We consider three cases.

Case 1.1 i and j are both isolated in B . Embed r to i , s to $i + 1$, q to j , and the children of q onto $[j - 1, i + 2]$. See Figure 15a. Note that i is not in edge-conflict due to the placement invariant. Every red edge is incident to i or j and hence does not occur in B by assumption.

Case 1.2 i is not isolated in B . If $B\langle i \rangle$ is a central-star, flip it if necessary to put its root (which is not in edge-conflict by the peace invariant) at i . Otherwise, use the leaf-isolation shuffle to put a leaf at $i + 1$ and its parent at i . Since $B\langle i \rangle$ is not a central-star, by Proposition 8, this will place the root of $B\langle i \rangle$ at some position $x > i + 1$. In both cases, embed r onto i , s onto j , q onto $i + 1$, and the children of q onto $[i + 2, j - 1]$. See Figure 15b. The edge $\{r, s\}$ is not used by B since $\{i, j\}$ is not used by assumption (and the leaf-isolation shuffle cannot change that). The red edges incident to q are not used since the only neighbor of $i + 1$ in B is i .

Case 1.3 i is isolated and j is not isolated in B . Flip $B\langle j \rangle$ if its root is currently at j . Note that $[j, i]$ is not in degree-conflict for embedding R : this would imply that B is a star since $\deg_R(r) = 1$. If $[j, i]$ is not in conflict, then the invariants hold for $[j, i]$ and we can apply Case 1.2 by embedding R on $[j, i]$ instead of $[i, j]$. Otherwise, $B\langle j \rangle$ is a central-star on at least two vertices.

If $|B\langle j \rangle| = 2$, then embed r onto j , s onto i , q onto $j - 1$, and the children of q onto $[j - 2, i + 1]$. See Figure 15c. If $|B\langle j \rangle| \geq 3$, then embed r onto j , s onto $j - 1$, q onto i , and the children of q onto $[i + 1, j - 2]$. See Figure 15d. This works because the root of $B\langle j \rangle$ is not at j (so j is

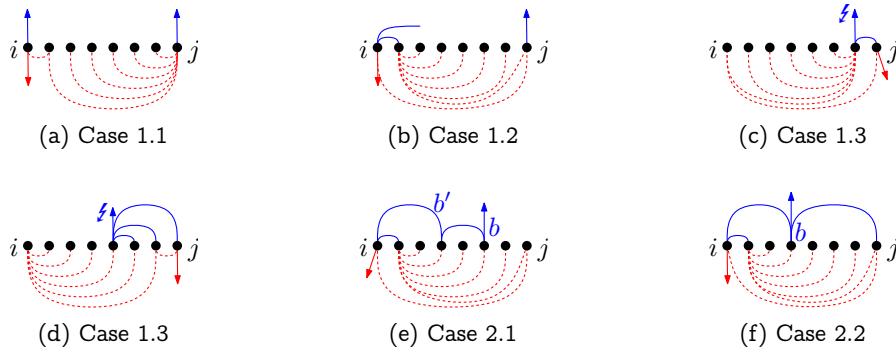


Figure 15: The case analysis in the proof of Proposition 11.

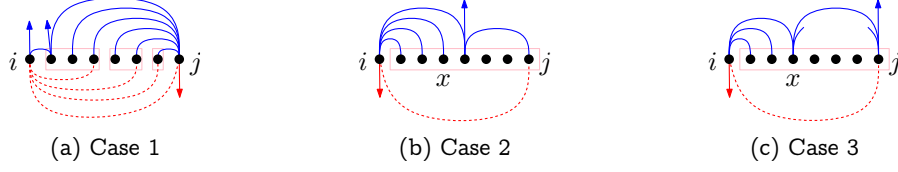


Figure 16: The case analysis in the proof of Proposition 12.

not in edge-conflict), the size of the star $B\langle j \rangle$ is at least three (so $\{j-1, j\}$ is not used), and i is isolated in B (so the red edges incident to q are not used by B).

Case 2 $\{i, j\} \in E(B)$. Let b be the root of B . We claim that (1) some vertex of B has distance at least three to b or (2) $\deg_B(b) \geq 2$. To prove the claim, suppose that all vertices in B have distance at most two to b and that b is unary. Then the child of b has distance one to all other vertices of B : hence B is a star centered at the child of b , a contradiction. We perform a case analysis on whether (1) or (2) holds.

Case 2.1 Some vertex v of B has distance at least three to b . Let b' be the child of b that contains v in its subtree B' . Let w be the size of B' . We re-embed B as follows. B' is not a central-star by choice of v . Hence, by Proposition 8, we can use the leaf-isolation shuffle to embed B' on $[i, i+w-1]$, placing a leaf at $i+1$, its parent at i , and the root b' at some position in $[i+2, i+w-1]$. Complete this embedding of B' to any one-page book embedding of B . Note that this embedding does not use the edge $\{i, j\}$. Embed r at i , s at j , q at $i+1$, and the children of q at $[i+2, j-1]$. See Figure 15e. This works because b is not at i (so i is not in edge-conflict), B does not use the edge $\{i, j\}$ (so $\{r, s\}$ is not used by B), and $i+1$ is isolated in $B[i+1, j]$ (so the red edges incident to q are not used by B).

Case 2.2 $\deg_B(b) \geq 2$. Since B is not a star, some vertex v has distance at least two to b in B . Let b' be the child of b that contains v in its subtree B' . Let w be the size of B' . We re-embed B as follows. Use the leaf-isolation shuffle to embed B' together with b on $[i, i+w]$, placing a leaf at $i+1$, its parent at i , and b at $i+w$. Complete this embedding to any one-page book embedding of B . Note that this embedding does not use the edge $\{i, j\}$. Finish by using the same embedding for R as in Case 2.1. See Figure 15f. \square

Proposition 12. *If $\deg_R(r) = 1$, S is not a star, and $\{i, j\} \in E(B)$, then there is an ordered plane packing of B and R onto I .*

Proof. Flip B if necessary to put its root at j . The general plan is to embed r onto i and S recursively onto $[i+1, j]$. This works unless (1) $B[i+1, j]$ is a star, (2) $\{i, i+1\} \in E(B)$, or (3) there is a conflict for embedding S onto $[i+1, j]$. Below, we find an ordered plane packing under a weaker condition than (1) to allow for reuse in cases (2) and (3). In case (2), by LSFR, $B[i, j-1]\langle i \rangle$ is a central-star on at least two vertices. In case (3), $B[i+1, j]\langle i+1 \rangle$ is a central-star. We deal with these cases below.

Case 1 $B[i+1, j]$ is a star or $B[i, j-1]$ is a star. If $B[i, j-1]$ is a star, then we flip B to reduce to the case that $B[i+1, j]$ is a star. Thus, in the following, assume that $B[i+1, j]$ is a star and that the root of B may be either at i or at j . We know exactly what B looks like: since B is not a star, the star $B[i+1, j]$ must be centered at $i+1$ and rooted at j . Flip the blue embedding at $[i+1, j]$: this puts the star-center at j . Note that $\{i, j\} \notin E(B)$. Embed r onto j . The interval $[i, j-1]$ is in edge-conflict with S if the root of B is now at $i+1$. Hence, we embed S explicitly. Embed s onto i . Since S is not a star, it must have a subtree of size $k \geq 2$. Embed this subtree explicitly at $[i+k, i+1]$. Embed the other subtrees of s explicitly on the remainder. See Figure 16a.

Case 2 $B[i, j-1]\langle i \rangle$ is a central-star on at least two vertices. Let x be such that $B[i, j-1]\langle i \rangle = B[i, x]$. By Case 1 we may assume that $x \leq j-2$. By LSFR at i and by choice of x , $B[x+1, j]$ is a tree. Flip $B[x+1, j]$. Since $x \leq j-2$, the root of B is no longer at j . Embed r onto i and S recursively onto $[j, i+1]$. See Figure 16b. Since $\{i, j\} \notin E(B)$ after flipping and $i+1$ is isolated in $B[i+1, j]$, this works unless $[j, i+1]$ is in conflict for S . Then $B[i+1, j]\langle j \rangle$ is a central-star that is rooted at the root of B . But this contradicts LSFR at j before flipping: a contradiction. Hence, there is no conflict for S .

Case 3 $B[i+1, j]\langle i+1 \rangle$ is a central-star. Let x be such that $B[i+1, x] = B[i+1, j]\langle i+1 \rangle$. Since $\{i, j\} \in E(B)$, $B[i+1, x]$ is rooted and centered at x and the parent of x is at i . Hence $B[i, x]$ is a dangling star. By Case 1 we may assume that $x \leq j-2$. Flip $B[i, x]$. Embed r at i and S recursively at $[j, i+1]$. See Figure 16c. Since $\{i, j\} \notin E(B)$ after flipping and $i+1$ is isolated in $B[i+1, j]$, this works unless $[j, i+1]$ is in conflict for S . Then $B[i+1, j]\langle j \rangle$ is a central-star that is rooted at the root of B . But this contradicts LSFR at j before flipping: a contradiction. Hence, there is no conflict for S . \square

Proposition 13. *If $\deg_R(r) = 1$, S is not a star, and $\{i, j\} \notin E(B)$, then there is an ordered plane packing of B and R onto I .*

Proof. The general plan is to embed r onto i and S recursively onto $[j, i+1]$. Since $\{i, j\} \notin E(B)$ and S is not a star, this works unless (1) $B[i+1, j]$ is a star or (2) there is a conflict for embedding S onto $[j, i+1]$. In case (2), the star $B^* := B[j, i+1]\langle j \rangle$ is either in edge-conflict or in degree-conflict for embedding S . If it is in edge-conflict, then there must be an edge from the root of B^* to r . By 1SR, the root of B^* must be at j . But that means that $\{i, j\} \in E(B)$, a contradiction. Thus, in case (2), there is a degree-conflict for embedding S onto $[j, i+1]$. We deal with these cases below.

Case 1 $B[i+1, j]$ is a star. Since $\{i, j\} \notin E(B)$, vertex i is isolated in B . Flip $B\langle j \rangle = B[i+1, j]$ if necessary to put the center of $B[i+1, j]$ at j . If the root of $B[i+1, j]$ is at $i+1$, then embed r onto j and S recursively onto the independent set $[i, j-1]$. Since i is isolated in the blue embedding, $[i, j-1]$ is not in conflict for S . If the root of $B[i+1, j]$ is at j , then flip the blue embedding at $[j-1, j]$. This places the root at $j-1$ and a leaf of the star at j . After flipping, the interval $[i, j-1]$ still satisfies the invariants. Embed r onto j (which is not in edge-conflict) and S recursively onto $[i, j-1]$. See Figure 17a. Since i is isolated in the blue embedding, $[i, j-1]$ is not in conflict for S .

Case 2 There is a degree-conflict for embedding S onto $[j, i+1]$. Let y be such that $B[j, i+1]\langle j \rangle = B[y, j]$. Due to the degree-conflict, $B[y, j]$ is a central-star on at least three vertices. Since $B[j, i+1]\langle j \rangle = B[y, j]$, the root of $B[y, j]$ is not adjacent to any vertex in $[i+1, y-1]$. By 1SR, if it were adjacent to i , then the root of $B[y, j]$ must be at j : this however, violates the assumption that $\{i, j\} \notin E(B)$. Hence, $B[y, j] = B\langle j \rangle$. Since $B[y, j]$ is a tree and $y \leq j-2$, $B[i, j-1]$ is not a star. We distinguish two cases.

Case 2.1 There is no conflict for embedding S onto $[i, j-1]$. Flip $B[y, j]$ if necessary to put its root at y . This preserves all invariants on $[i, j]$. Embed r onto j (which is not the root of $B[y, j]$)

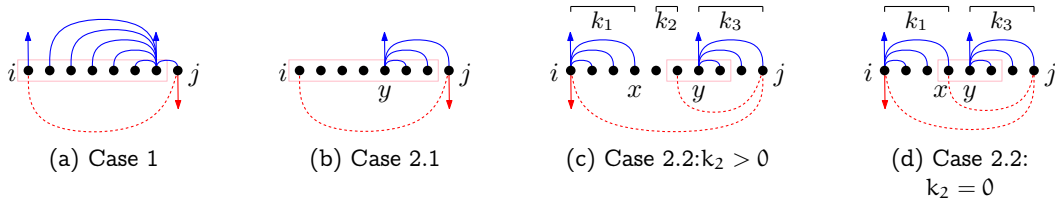


Figure 17: The case analysis in the proof of Proposition 13.

and S recursively onto $[i, j - 1]$. See Figure 17b. This works by the assumption that there is no conflict for embedding S onto $[i, j - 1]$ before flipping $B[y, j]$ and since $B\langle i \rangle \neq B\langle j \rangle$ due to $\{i, j\} \notin E(B)$.

Case 2.2 There is a conflict for embedding S onto $[i, j - 1]$. By the 1SR and the fact that $\{i, j\} \notin E(B)$, there is a degree-conflict for embedding S onto $[i, j - 1]$. Let x be such that $B[i, j - 1]\langle i \rangle = B[i, x]$. By the same argumentation that proved $B[y, j] = B\langle j \rangle$ we have $B[i, x] = B\langle i \rangle$. Thus, we can divide B into three disjoint parts: $B[i, x]$ (a central-star), $B[x + 1, y - 1]$ (about which we know nothing), and $B[y, j]$ (a central-star). For notational convenience, let $k_1 = |[i, x]|$, $k_2 = |[x + 1, y - 1]|$, and $k_3 = |[y, j]|$ be the corresponding sizes. Let $d = \deg_S(s)$ and let v_1, \dots, v_d be the children of s , ordered by increasing size of their subtrees ($t_S(v_d)$ is the largest). Since S is not a star $|t_S(v_d)| \geq 2$. Let λ be the number of leaf children of s . Then $|t_S(v_\ell)| = 1$ if and only if $\ell \leq \lambda$.

Flip $B[i, x]$ if necessary to put the root (and center) at i and flip $B[y, j]$ if necessary to put the root (and center) at y . We first explain how to embed R and then prove that it always works. Refer to Figure 17c for the case $k_2 > 0$ and Figure 17d for the case $k_2 = 0$. Embed r onto i and s onto j . This works so far: by the peace invariant the root of $B[i, x] = B\langle i \rangle$ is not in conflict and $\{i, j\} \notin E(B)$. Next, embed $t(v_d)$ recursively onto $[y - 1, y + |t(v_d)| - 2]$. Since $\{y - 1, j\} \notin E(B)$ and $y - 1$ is isolated in $B[y - 1, y + |t(v_d)| - 2]$, this works provided $t(v_d)$ fits inside $[y - 1, j - 1]$, i.e. provided $|t(v_d)| \leq |[y - 1, j - 1]| = |[y, j]| = k_3$. Next, embed a leaf child of s on each vertex in $[x + 1, y - 2]$ (this interval may be empty). This embeds the children v_1, \dots, v_{k_2-1} and works provided that $\lambda \geq k_2 - 1$. This leaves two disjoint intervals to embed the remaining subtrees $t(v_{\max(1, k_2)}), \dots, t(v_{d-1})$ of s : $I_1 := [i + 1, \min(x, y - 2)]$ and $I_2 := [y + |t(v_d)| - 1, j - 1]$. Thus, it remains to prove that (i) $|t(v_d)| \leq k_3$, and (ii) $\lambda \geq k_2 - 1$, and that (iii) we can distribute the remaining subtrees over I_1 and I_2 .

We begin by showing that d must be large. Since there is a degree-conflict for embedding S onto $[j, i + 1]$ we have $k_1 + k_2 - 1 < d$, and since there is a degree-conflict for embedding S onto $[i, j - 1]$ we have $k_2 + k_3 - 1 < d$:

$$k_1 + k_2 \leq d; \quad (7)$$

$$k_2 + k_3 \leq d. \quad (8)$$

Recall that $k_1 + k_2 + k_3 = |R| = |S| + 1$. Adding (7) and (8) yields $2d \geq k_1 + k_2 + k_3 + k_2 = |S| + 1 + k_2$ and so

$$d \geq \frac{|S| + 1 + k_2}{2}, \quad (9)$$

Proof of (i) We must show that $|t(v_d)| \leq k_3$. Using (7) we get $\sum_{\ell=1}^{d-1} |t(v_\ell)| \geq d - 1 \geq k_1 + k_2 - 1 = |S| - k_3$. Since the total size of the subtrees at the children of S is $|S| - 1$ we have $|t(v_d)| = |S| - 1 - \sum_{\ell=1}^{d-1} |t(v_\ell)| \leq |S| - 1 - |S| + k_3 = k_3 - 1 < k_3$, which completes the proof of (i).

Proof of (ii) We must show that $\lambda \geq k_2 - 1$. Since $|t(v_\ell)| \geq 2$ for all ℓ , $\lambda + 1 \leq \ell \leq d$, we have $2(d - \lambda) + \lambda \leq |S| - 1$ and so

$$\lambda \geq 2d - |S| + 1 \stackrel{(9)}{\geq} (|S| + 1 + k_2) - |S| + 1 = k_2 + 2.$$

Proof of (iii) It remains to prove that we can distribute $t(v_{\max(1, k_2)}), \dots, t(v_{d-1})$ over the disjoint intervals $I_1 = [i + 1, \min(x, y - 2)]$ and $I_2 = [y + |t(v_d)| - 1, j - 1]$. We use the following observation on partitioning natural numbers.

Observation 14. Let n and t be positive integers with $t \geq \lfloor n/2 \rfloor + 1$ and let $a_1 \leq \dots \leq a_t$ be positive integers with $\sum_{i=1}^t a_i = n$. Then for all $0 \leq k \leq n$ there exists a set $J_k \subseteq [1, t]$ such that $\sum_{i \in J_k} a_i = k$.

Proof. We prove the statement by induction on n . The statement is true for $n = 1$: in this case we must have $t = 1$ and $a_1 = 1$, and so $J_0 = \emptyset$ and $J_1 = \{1\}$ work. Suppose that the statement holds for all positive integers smaller than n . It suffices to prove the statement for $k \geq \lceil n/2 \rceil$ since we can choose $J_k = [1, t] \setminus J_{n-k}$ for $k < \lceil n/2 \rceil$. If $a_t = 1$ then $a_1 = \dots = a_t = 1$ and we choose $J_k = [1, k]$. Otherwise, by the assumption on t we have $a_t = n - \sum_{i=1}^{t-1} a_i \leq n - t + 1 \leq \lceil n/2 \rceil$ and hence $k - a_t \geq 0$. By the assumption on t and since $a_t \geq 2$ we have $t - 1 \geq \lfloor n/2 \rfloor \geq \lfloor (n - a_t)/2 \rfloor + 1$. Hence, by the induction hypothesis, there exists a set $J_{k-a_t} \subseteq [1, t-1]$ with $\sum_{i \in J_{k-a_t}} a_i = k - a_t$. Choose $J_k = J_{k-a_t} \cup \{t\}$ to complete the proof. \square

The total size of the remaining subtrees is $n := |S| - 1 - \sum_{\ell=1}^{k_2-1} |t(v_\ell)| - |t(v_d)| \leq |S| - 1 - \max(0, k_2 - 1) - 2 = |S| - 2 - \max(1, k_2)$ since $|t(v_d)| \geq 2$. Then

$$t := d - 1 \stackrel{(9)}{\geq} \frac{|S| + 1 + k_2}{2} - 1 = \frac{|S| - 3 + k_2}{2} + 1 \geq \frac{n}{2} + 1,$$

where the last step uses that $|S| - 3 + k_2 \geq |S| - 2 - k_2$ for $k_2 \geq 1$ and $|S| - 3 + k_2 \geq |S| - 2 - 1$ for $k_2 = 0$. Hence, n and t satisfy the precondition of Observation 14. We apply the observation with $k = |I_1|$. This gives us a set J_k such that $\sum_{\ell \in J_k} |t(v_\ell)| = |I_1|$ and $\sum_{\ell \in [1, d-1] \setminus J_k} |t(v_\ell)| = |I_2|$.

Since $B[I_1]$ and $B[I_2]$ have no internal edges and no edges to the position of r at j , we can embed the subtrees $t(v_\ell)$ with $\ell \in S_k$ explicitly from left to right on I_1 and the remaining subtrees explicitly from left to right on I_2 . This completes the proof. \square

Proposition 11, Proposition 12, and Proposition 13 together prove the following.

Lemma 15. *If $\deg_R(r) = 1$, then there is an ordered plane packing of B and R onto I .*

8 Embedding the red tree: a singleton subtree

Here we completely handle the case $|S| = 1$.

Lemma 16. *If $|S| = 1$, then R and B admit an ordered plane packing onto $[i, j]$.*

Proof. We distinguish two cases.

Case 1 R^- is not a star. We first describe an embedding that works whenever $B[i, j-1]$ is a star. Flip $B[i, j-1]$ if necessary to put its center at $j-1$. In addition to the star at $[i, j-1]$, the blue embedding may use the edge $\{i, j\}$. Note that it cannot use $\{j-1, j\}$, as this would imply that B is a star. Thus, j is isolated in $B[i+1, j]$. Embed r onto $i+1$ and s onto i . Let U be a largest subtree of r in R^- . Since R^- is not a star, $|U| \geq 2$. Embed U recursively onto $[j, j-|U|+1]$. Since j is locally isolated in $B[j, j-|U|+1]$ and j is not adjacent to $i+1$ (which is where we embedded r), this always works. Embed the remaining subtrees of r in R^- explicitly on $[i+2, j-|U|]$.

Assume now that $B[i, j-1]$ is not a star. If $\{i, j\} \notin E(B)$, then we embed s at j , and recursively embed R^- onto $[i, j-1]$. R^- has no edge-conflict with $[i, j-1]$ by the peace invariant. It also has no degree-conflict with $[i, j-1]$: otherwise R would already have had a degree-conflict with $[i, j]$.

So assume that $\{i, j\} \in E(B)$. Flip B if necessary to put its root at j . If $B[i, j-1]$ is a star now, then use the embedding described in the first paragraph to find an ordered plane packing. Otherwise, r is not in edge-conflict with any vertex in $[i, j-1]$. The general plan is to embed s at j and R^- recursively onto $[j-1, i]$. Since B is not a star and B is rooted at j , the edge $\{j-1, j\}$ is not used. Hence, this works unless there is a conflict for embedding R^- onto $[j-1, i]$. This means in particular that $B[i, j-1] \setminus \{j-1\}$ is a central-star $B^* = B[x, j-1]$. See Figure 18a. By assumption, $i+1 \leq x \leq j-2$. Due to the presence of the edge $\{i, j\}$ and since $x \geq i+1$, the root (and hence also the center) of B^* must be at x .

Case 1.1 $x \geq i + 2$. Flip $B[x, j]$. Note that afterwards $\{i, j\} \notin E(B)$ and $B[i, j - 1]$ satisfies 1SR and LSFR. Embed s onto j and R^- recursively onto $[i, j - 1]$. See Figure 18b. Since $|B^*| \geq 2$, the interval $[i, j - 1]$ contains at least one leaf of B^* and so $B[i, j - 1]$ is not a star. Hence, this works unless there is a conflict for embedding R^- onto $[i, j - 1]$. In that case, note that $B[i, x]\langle i \rangle$ is now formed by the root of B and its subtrees other than B^* . Since $B[i, x]\langle i \rangle$ is a central-star, it follows that the subtrees of the root b of B other than B^* are all leaves. Flip $B[x, j]$ again to restore the original embedding. Embed s onto i and R^- recursively onto $[i + 1, j]$. See Figure 18c. Since $x \geq i + 2$, $B[i + 1, j]$ is a tree that is not a star and $\{i, i + 1\} \notin E(B)$. Hence, the peace invariant holds for R^- .

Case 1.2 $x = i + 1$. Flip $B[x, j]$. Embed r onto i and s onto j . Embed the remaining subtrees of r in R explicitly onto the independent set $B[i + 1, j - 1]$, putting the largest one (which has size at least two) next to i . See Figure 18d.

Case 2 R^- is a star. Then $\deg_R(r) = 2$ and the child q of r in R^- is the root and center of a star $Q = t(q)$.

Case 2.1 $\{i, j\} \in E(B)$. Let b be the root of B . If $\deg_B(b) = 1$, then flip B if necessary to put its root at j . Then j is isolated in $B[i + 1, j]$ and $\{i, i + 1\} \notin E(B)$ since B is not a star and by LSFR. Embed r onto $i + 1$, s onto i , q onto j , and the children of q onto $[j - 1, i + 2]$. See Figure 19a.

If $\deg_B(b) \geq 2$, then flip B if necessary to put its root at j . Let x be such that $B[i, j - 1]\langle i \rangle = B[i, x]$, which is a smallest subtree of b . Since B is not a star, $B[x + 1, j]$ is not a central-star. Flip $B[x + 1, j]$. This puts the root b at $x + 1$. Use a leaf-isolation-shuffle on $B[x + 1, j]$ to embed a leaf at $j - 1$, its parent of j , and the root at $x + 1$. This works by Proposition 8. Embed r onto i , s onto j , q onto $j - 1$ and the children of q onto $[j - 1, i + 1]$. See Figure 19b.

Case 2.2 $\{i, j\} \notin E(B)$. Then $B\langle i \rangle \neq B\langle j \rangle$. If $|B\langle j \rangle| \geq 2$, then perform a leaf-isolation-shuffle to put a leaf at $j - 1$ and its parent at j . Since $B\langle i \rangle \neq B\langle j \rangle$, this does not touch the blue vertex at i . Embed r onto i , s onto j , q onto $j - 1$, and the children of q onto $[j - 2, i + 1]$. See Figure 19c.

If $|B\langle j \rangle| = 1$ and $B\langle i \rangle$ is not a central-star, then flip $B\langle i \rangle$ if necessary to put its root at i . Since it is not a central-star, $\{i, i + 1\} \notin E(B)$. Embed r onto $i + 1$, s onto i , q onto j , and the children of q onto $[j - 1, i + 2]$. See Figure 19d.

Finally, if $|B\langle j \rangle| = 1$ and $B\langle i \rangle$ is a central-star, then let x such that $B[i, x] = B\langle i \rangle$. We have $x \leq j - 2$ by the peace invariant. Flip $B[i, x]$ if necessary to put its root at i . By the peace invariant, i is not in edge-conflict with r . Embed r onto i , s onto $x + 1$, q onto j , and the children of q onto $[j - 1, x + 2]$ and $[x, i + 1]$. See Figure 19e. \square

9 Embedding the red tree: a large blue star

In this and the following section we handle the case that $B[i, j - |S|]$ is a star. The graphs S , R^- , and $B[j - |S| + 1, j]$ may or not be stars. The case that we actually handle is more general, as specified in the following

Lemma 17. *If $B[i, x]$ is a star, for $x \in [j - |S|, j - 1]$, then R and B admit an ordered plane packing onto $[i, j]$.*

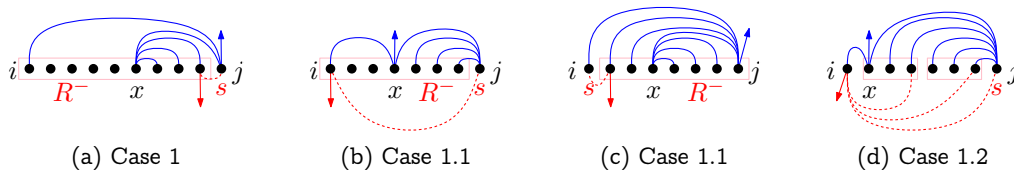


Figure 18: Case 1 in the proof of Lemma 16.

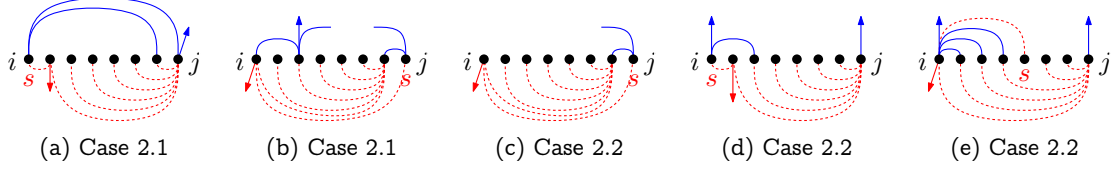


Figure 19: Case 2 in the proof of Lemma 16.

Proof. By Lemma 15 and Lemma 16, we may assume $\deg_R(r) \geq 2$ and $|S| \geq 2$. The following observation does not depend on the context of this proof.

Observation 18. $|R^-| \neq 2$.

Proof. If $|R^-| = 2$, then by the minimality of S we have $|S| = 1$. It follows that $|R| = 3$ and so R is a star, contrary to our assumption. \square

By Observation 18, $|R^-| \geq 3$. Select x maximally so that $B^* = B[i, x]$ is a star, and let $d = \deg_{R^-}(r) \geq 1$. Note that $|B[i, x]| \geq |R^-| \geq 3$. We distinguish two cases.

Case 1 B^* is a central-star. Then by LSFR we have $B\langle i \rangle = B^*$. If necessary, flip B^* to put its root and center at i . We will use a blue-star embedding to embed R^- from $\sigma = i$ with $\varphi = (x+1, \dots, x+d)$. Let us first check the conditions for the blue-star embedding. (BS1) holds by (I1) for embedding R onto $[i, j]$. For (BS2) we must show $|R^-| \leq |B^*| + d$ and $|B^*| + d \leq |I| - 1$. We wish to argue that at least one leaf of B^* remains after the blue-star embedding, and thus we show $|R^-| < |B^*| + d$. This inequality holds since $x \geq j - |S|$ and $d \geq 1$. For the second inequality, by (I1), we have $|B^*| \leq |I| - \deg_R(r)$ and so $|B^*| + d = |B^*| + \deg_R(r) - 1 \leq |I| - 1$. (BS3) and (BS4) hold since $B \setminus (B^* \cup \varphi)$ forms an interval. Hence, by Proposition 6, the blue-star embedding succeeds and leaves an interval $[i', j'] = [i', j]$ such that j is not in edge-conflict for embedding s and a non-empty prefix of $[i', j]$ consists of isolated vertices that are in edge-conflict for embedding s (these are leaves of B^*). Recursively embed S onto $[j, i']$. This works unless S is a star or $B[i', j]\langle j \rangle$ is in conflict (which must be a degree-conflict) for embedding S .

Case 1.1 S is a star. If S is a dangling star then embed s onto j , the child s' of s onto i' (which is locally isolated), and the children of s' onto $[i' + 1, j - 1]$. Otherwise, S is a central-star. If there is a locally isolated vertex in $B[i', j]$ that is not in edge-conflict, then use this vertex to embed s and embed the children of s on the remainder. Otherwise, undo the blue-star embedding. Consider the blue vertex at j . It does not get consumed by the blue-star embedding. Since it was not isolated after the blue-star embedding, it is not isolated now. By choice of x , we have $B\langle j \rangle = B[x + 1, j]\langle j \rangle$. Perform a leaf-isolation-shuffle on $B\langle j \rangle$ to place a leaf ℓ at $j - 1$ and its parent at j . Perform the original blue-star embedding, but now with $\varphi = (j, x + 1, \dots, x + d - 1)$ if $d \geq 2$ and $\varphi = (j)$ if $d = 1$. The conditions of the blue-star embedding still hold. The resulting interval $[i', j']$ contains the now isolated vertex ℓ and we embed S by placing s onto ℓ and embedding the children of s on the remainder.

Case 1.2 $B[i', j]\langle j \rangle$ is a central-star that raises a degree-conflict. Note that $[i', j]$ is composed of some locally isolated vertices plus some a suffix of the interval $[i, j]$ before the blue-star embedding. Undo the blue-star embedding. Now $B[z, j]$ is a central-star for some minimal z . We claim that we may assume that $B[z, j]$ is rooted at j . Indeed, if $B[z, j]$ is rooted at z , then by 1SR we have $B[z, j] = B\langle j \rangle$ and we can flip $B\langle j \rangle$ to establish the claim. Perform the original blue-star embedding for R^- , but now with $\varphi = (j, x + 1, \dots, x + d - 1)$ if $d \geq 2$ and $\varphi = (j)$ if $d = 1$. In the remaining interval $[i', j']$, the vertex j' is a leaf of what was the central-star $B[z, j]$ before the blue-star embedding. Hence, j' is locally isolated and not in edge-conflict with s . Recursively embed S onto $[j', i']$ to complete the embedding.

Case 2 B^* is a dangling star. In this case (I1) does not tell us anything about the size of B^* (because it applies to central-stars only). If the root of B^* is at x , then its center is at i and by 1SR i is the only neighbor of x in B . Hence by flipping B^* we may suppose that the root of B^* is at i . Note that i may have more neighbors, in addition to the center of B^* at x . Also note that i may be in conflict with r , in case we flipped B^* (the original vertex at i cannot be in conflict by (I3)). We distinguish two cases.

Case 2.1 $x = j - 1$. In this case we know almost completely what B looks like: $B[i, j - 1]$ is a star rooted at i and centered at $j - 1$ and the edge $\{i, j\}$ may or may not be used. We embed R explicitly as follows. Since $\deg_R(r) \geq 2$, there is a subtree $W = t(w)$ of r different from S . Embed r onto $i + |W|$ and embed W explicitly onto the independent set at $[i, i + |W| - 1]$. Since $|S| \geq 2$, we know that $|R'| < |B[i, j - 1]|$, and hence r is not embedded at the center of the star $B[i, j - 1]$. If S is not a star, embed it recursively onto $[j, j - |S| + 1]$. This works because j is locally isolated in $B[j - |S| + 1, j]$ and j is not adjacent to $i + |W|$ (which is where we embedded r). If S is a central-star, embed s onto j and its children onto $[j - 1, j - |S| + 1]$. If S is a dangling star, embed s onto $j - |S| + 1$, the child s' of s onto j , and the children of s' onto $[j - 1, j - |S| + 2]$. Embed the remaining subtrees (if any) of r on the remaining interval $[i + |W| + 1, j - |S|]$, which forms a locally independent set, none of whose vertices are adjacent to r .

Case 2.2 $x \leq j - 2$ and $B[j, j - |S| + 1]\langle j \rangle$ is a central-star B^{**} on $|B^{**}| \geq |S| - \deg_S(s) + 1$ (in particular, this holds if S has a degree-conflict for embedding on $[j, j - |S| + 1]$). We distinguish two subcases.

Case 2.2.1 $\{i, j\} \in E(B)$. Then the root and center b^{**} of B^{**} must be at j and cannot be the root of B because then LSFR would imply that B is a star. Therefore i is the root of B (Figure 20a) and it is not in conflict with r due to (I3). We modify the embedding of B as follows: Move b^{**} to $j - |S|$ and all leaves of B^{**} (as $|B^{**}| \geq |S| - \deg_S(s) + 1 \geq 2$, there is at least one) in sequence immediately to the right of b^{**} , at position $j - |S| + 1$ and onward, shifting all vertices between there and j to the right accordingly. Draw the edge $\{i, b^{**}\}$ below the spine to avoid crossings, and all other edges incident to b^{**} above the spine (Figure 20b).

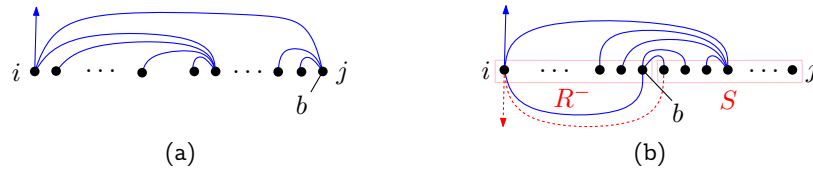


Figure 20: $\{i, j\} \in E(B)$ (Case 2.2.1).

We place r at i and explicitly embed R^- onto $[i, j - |S|]$, which in B consists of a single edge $\{i, j - |S|\}$ with isolated vertices (at least one because $|R^-| \geq 3$) in between. Recall that R^- is not a central-star and so we can embed it as described. It remains to embed S onto $[j - |S| + 1, j]$. As $j - |S| + 1$ is a leaf of B^{**} , which is isolated on $[j - |S| + 1, j]$, there is no conflict for this embedding and $B[j - |S| + 1, j]$ is not a star. Therefore, if S is not a star, then we can complete the packing recursively by embedding S onto $[j - |S| + 1, j]$.

It remains to consider the case that S is a star. If S is a central-star, then we can put this center at the locally isolated vertex $j - |S| + 1$. Otherwise, S is a dangling star with $|S| \geq 3$. As $|B^{**}| \geq |S| - \deg_S(s) + 1 = |S| \geq 3$, we have at least two locally isolated vertices (leaves of B^{**}) at $j - |S| + 1$ and $j - |S| + 2$. We put the root of S at $j - |S| + 1$ and the center at $j - |S| + 2$ to complete the packing.

Case 2.2.2 $\{i, j\} \notin E(B)$. Let us consider the central-star $B^{**} = B[j, j - |S| + 1]\langle j \rangle$. We claim that $B^{**} = B\langle j \rangle$. Indeed, if the root b^{**} of B^{**} is on the left, then by 1SR $B^{**} = B\langle j \rangle$. Otherwise, b^{**} is at j . By definition of B^{**} , b^{**} has no neighbors in $B[j, j - |S| + 1] \setminus B^{**}$. Since $B^* = B[i, x]$

is a star and $x \geq j - |S|$, the only remaining possible neighbor of b^{**} would be i , but this is excluded by the assumption. We conclude that $B^{**} = B(j)$. If necessary, flip B^{**} to put its root (and center) at j .

Case 2.2.2.1 $x = j - |S|$ (Figure 21a). Then we change the embedding of B by moving one leaf ℓ of B^{**} all the way to the left at i . As a leaf of B^{**} it is not in conflict with r , and so we can map r to $\ell = i$ and embed R^- explicitly onto the locally independent set $B[i, j - |S|]$. If S is not a star, then we recursively embed S onto $[j - |S| + 1, j]$ (Figure 21b). Note that $j - |S| + 1$ is the center of B^* , which is isolated in $B[j - |S| + 1, j]$ and not adjacent to the leaf of B^{**} at i . Therefore, $B[j - |S| + 1, j]$ is not a star and there is no degree-conflict and no edge-conflict for embedding S onto $[j - |S| + 1, j]$.

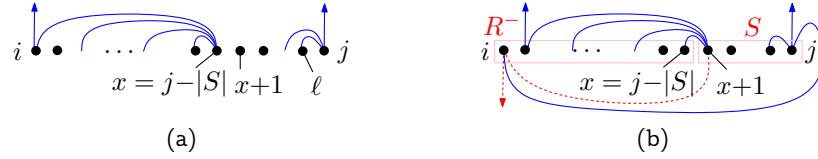


Figure 21: $x = j - |S|$ (Case 2.2.2.1).

It remains to consider the case that S is a star. If S is a central-star, then the center can be embedded on the isolated vertex at $j - |S| + 1$. Otherwise, S is a dangling star with $|S| \geq 3$. Then at least one more leaf of B^{**} remains at $j - 1$, where we can embed the root of S . The center of S is again mapped to the isolated vertex $j - |S| + 1$ and the edge $\{j - |S| + 1, j\}$ is drawn as a biarc, crossing the spine between $j - 2$ and $j - 1$.

Case 2.2.2.2 $x \geq j - |S| + 1$ (Figure 22a). Then we change the embedding of B by simultaneously moving the root of B^* to x and moving all other vertices of B^* to the left by one (Figure 22b). Embed r at i . We will use a blue-star embedding to embed S on $B[i + 1, j]$ from $\sigma = j$ where φ consists of the rightmost $\deg_S(s)$ non-neighbors of j in B from right to left. Note that $B[i + 1, j](j) = B^{**}$ and hence $B^* = B^+$ in the terminology of the blue-star embedding. Let us check the conditions for the blue-star embedding. (BS1) holds because $|R^-| \geq 3$ and so i is a leaf of B^* that is adjacent to $x - 1 \neq j$ only. (BS3) and (BS4) hold because $B[i + 1, j] \setminus (B^{**} \cup \varphi)$ forms an interval. For (BS2) we must show $|S| \leq |B^{**}| + \deg_S(s)$ and $|B^{**}| + \deg_S(s) \leq |I| - 2$. The first inequality follows from the assumption of Case 2.2. For the second inequality, we have $|B^{**}| \leq |B[x + 1, j]| \leq |S| - 1$ and $\deg_S(s) \leq |S| - 1$, and so $|B^{**}| + \deg_S(s) \leq 2|S| - 2 < |I| - 2$, since $|S| < |I|/2$. Hence, the conditions for the blue-star embedding are satisfied.

Since $\deg_S(s) \geq |S| - |B^{**}| + 1$, we have $\varphi \supset B[j - |S|, j] \setminus B^{**}$, and hence the blue-star embedding embeds a child of s onto the root of B^* , which was embedded at x , and the center of B^* , which was embedded at $x - 1$. Therefore, the remaining vertices not used for the embedding of S form an independent set in B and we can explicitly embed R^- on them.

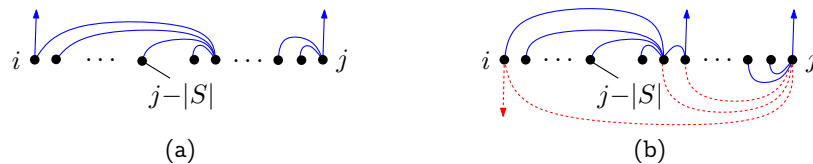


Figure 22: $x \geq j - |S| + 1$ (Case 2.2.2.2).

Case 2.3 $x \leq j - 2$ ($\Rightarrow |S| \geq 2 \Rightarrow |R| \geq 5$) and $B[j, j - |S| + 1](j)$ is **not** a central-star B^{**} on $|B^{**}| \geq |S| - \deg_S(s) + 1$ vertices. We first prove a claim and then distinguish two subcases.

Claim: We may suppose that S is a star or $x = j - |S|$. To prove the claim, suppose that

$x \geq j - |S| + 1$. Then $j - |S|$ is a leaf of B^* and we can explicitly embed R^- onto the independent set $[j - |S|, i]$. As $x \neq j$ is the only neighbor of $j - |S|$ in B , we have $\{j - |S|, j\} \notin E(B)$ and so there is no edge-conflict for embedding S onto $[j, j - |S| + 1]$. By assumption there is no degree-conflict for this embedding, either, and $B[j - |S| + 1, j]$ is not a star because the root of B^* is part of it but $B^* \neq B$. The only remaining obstruction for the recursive embedding of S onto $[j, j - |S| + 1]$ is S being a star. This proves the claim.

Case 2.3.1 $\{i, x + 1\} \notin E(B)$. Then we rearrange the embedding of B as follows: move the center c of B^* to $j - |S| + 1$ and the vertex b' at $x + 1$ (the leftmost vertex not in B^*) to $j - |S|$. In order to avoid crossings with the edge(s) incident to b' , draw all edges between c and its neighbors in $[i, j - |S| - 1]$ below the spine, whereas edges to neighbors in $[j - |S| + 2, j]$ remain above the spine (Figure 23). After this transformation $B[i, j - |S|]$ is an independent set, on which we can embed R^- explicitly. However, we have to take care because of the blue edges drawn below the spine and the (possibly) conflicting root i . Without loss of generality suppose that the root of B^* at i is in conflict with r .

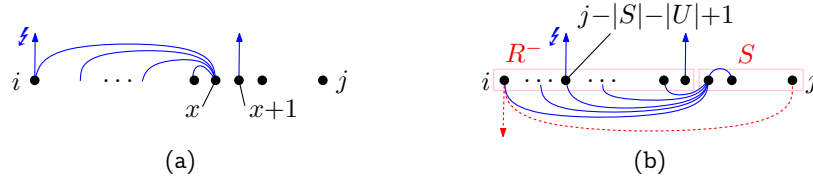


Figure 23: $x \leq j - 2$ and $\{i, x + 1\} \notin E(B)$ (Case 2.3).

Recall that R^- is not a central-star and so there is at least one non-leaf child u of r in R^- . Denote $U = t(u)$ and map both u and the conflicting root of B^* to $j - |S| - |U| + 1$ (by exchanging the order of leaves of B^* in $B[i, j - |S| - 1]$). As $|U| \geq 2$, we have $j - |S| - |U| + 1 \leq j - |S| - 1$ and so the local order for the roots of subtrees from B is maintained. On the other hand, we have $j - |S| = i + |R^-| - 1$ and $|U| \leq |R^-| - 1$, which imply $j - |S| - |U| + 1 \geq (i + |R^-| - 1) - (|R^-| - 1) + 1 = i + 1$. Therefore (the leaf now at) i is not in conflict with r .

As $\{i, x + 1\} \notin E(B)$ initially, after the transformation we have $\{j - |S| - |U| + 1, j - |S|\} \notin E(B)$ and so $[j - |S| - |U| + 1, j - |S|]$ is an independent set in B . Therefore, we can embed U onto $[x - |U| + 1, x]$ explicitly, drawing all edges above the spine, and complete the embedding of R^- by embedding $R^- \setminus U$ onto $[i, x - |U|]$ explicitly, again drawing all edges above the spine. After these changes to the embedding of B , the only neighbor of i in B is $j - |S| + 1$. Together with $|S| \geq 2$ it follows that there is no edge-conflict for recursively embedding S onto $[j, j - |S| + 1]$. We also know that $B[j, j - |S| + 1]$ is not a star because it contains part of B^* (at least the center at $j - |S| + 1$) and at least one more vertex not connected to that part of B^* : the vertex at j . (As $|S| \geq 2$, there were at least two such vertices initially, but one, the vertex b' , has been moved and used for embedding R^- .) Two possible obstructions for the recursive embedding of S onto $[j, j - |S| + 1]$ remain: a degree-conflict or S is a star. We conclude by considering both cases.

Case 2.3.1.1 S is a star. Undo the rearrangement of B^* . We will redo the rearrangement, but first do some other modifications.

Suppose first that $|B[x + 1, j] \setminus \{x + 1\}| \geq 2$ or $|B[x + 1, j] \setminus \{x + 2\}| = 1$. In the former case, use a leaf-isolation shuffle on $B[x + 1, j]$ to place a leaf at $x + 2$ and its parent at $x + 1$. We can apply the shuffle because $x \leq j - 2$ and therefore $|B[x + 1, j]| \geq 2$. After the modification (as described in the first paragraph of Case 2.3.1) we proceed as follows. If S is a central-star, then s can be placed at $x + 2$, which is adjacent to $j - |S|$ only and therefore locally isolated in $B[j - |S| + 1, j]$. Otherwise, S is a dangling star. Then either there is a (non-root) leaf of B^* in $[j - |S| + 1, j]$ or the center c of B^* is isolated in $[j - |S| + 1, j]$. In either case, we put the center of S on $x + 2$. In the former case, we put the root of S on $j - |S| + 2$ (the leftmost leaf of B^* in $[j - |S| + 1, j]$),

and draw the edge $\{c, x+2\}$ as a biarc that crosses the spine between $j - |S| + 2$ and $j - |S| + 3$. In the latter case, we put the root of S on $j - |S| + 1 = c$. Either way, we can complete the star S and the embedding of R^- works just as before.

Otherwise, $|B[x+1, j]\langle x+1 \rangle| = 1$ and $|B[x+1, j]\langle x+2 \rangle| \geq 2$. Since $\{i, x+1\} \notin E(B)$ by the assumption of Case 2.3.1, we know that $B\langle x+1 \rangle = B[x+1, j]\langle x+1 \rangle$ and hence also $B\langle x+2 \rangle = B[x+1, j]\langle x+2 \rangle$. It follows that $x \leq j - 3$ and hence $|S| \geq 3$. Perform a leaf-isolation-shuffle on $B\langle x+2 \rangle$ to place a leaf at $x+3$ and its parent at $x+2$. Rearrange the embedding of B as follows: move the center c of B^* to $j - |S| + 1$, the vertex b' at $x+1$ to $j - |S| - 1$, and the vertex b'' at $x+2$ to $j - |S|$. Draw the blue edges as explained in the first paragraph of Case 2.3.1. We embed S analogously to the previous paragraph, using $x+3$ as the location for the star-center. To embed R^- , let us consider the embedding $B[i, j - |S|]$. It is again an independent set. As opposed Case 2.3.1, however, we have local roots at $j - |S| - 1$ and at $j - |S|$. Fortunately, since $|S| \geq 3$ and S is a smallest subtree, also $|U| \geq 3$, and hence $j - |S| - |U| + 1 \leq j - |S| - 2$, as required. Hence, we can embed R^- explicitly, analogously to the second paragraph of Case 2.3.1.

Case 2.3.1.2 There is a degree-conflict for embedding S onto $[j, j - |S| + 1]$. Then this conflict must have been created by our transformation of the embedding of B . Before this transformation there was no degree-conflict by assumption (Case 2.2 handles this scenario). In other words, b' is the root of a star $B[x+1, j]$ in the initial embedding whose center is at j . After moving b' out of the interval $[j - |S| + 1, j]$, j became the local root, which raised the degree-conflict. By our claim and the preceding Case 2.3.1.1 we may suppose that $x = j - |S|$ (Figure 24a). We use a different, explicit embedding as follows: flip the star $B[x+1, j]$ so that its root is at j and the center is at $x+1$ and draw all edges below the spine. Next move the center at $x+1$ to i instead, shifting all vertices in between to the right by one. Then put r at i (not being the root of $B[x+1, j]$ it is not in conflict), and explicitly embed R^- onto the (now) independent set $[i, x]$. Finally, explicitly embed S onto the (now) independent set $[x+1, j]$ (Figure 24b). Note that B might be a tree (in which case the two roots in the figure are actually connected), but the embedding works also in this case.

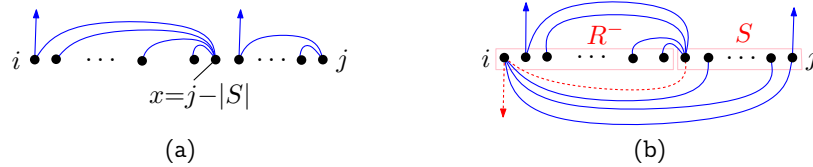


Figure 24: A new degree-conflict for S on $[j, j - |S| + 1]$ (Case 2.3.1.2).

Case 2.3.2 $\{i, x+1\} \in E(B)$ and $x \geq j - |S| + 1$. Then by our claim we may suppose that S is a star. We embed R^- explicitly onto $[j - |S|, i]$, noting that $j - |S|$ is a non-root leaf of B^* and, therefore, not in conflict with r . If S is a central-star, then we put the center at $x+1$, which is connected to i only and therefore locally isolated on $[j - |S| + 1, j]$. Otherwise, $|S| \geq 3$ and S is a dangling star. Given that $x \leq j - 2$, we have $x+1 \neq j$ and therefore can put the root of S on j and the center on $x+1$.

Case 2.3.3 $\{i, x+1\} \in E(B)$ and $x = j - |S|$. We distinguish two final subcases.

Case 2.3.3.1 The root of $B[i, x+1]$ is at i . Then we change the embedding of B by moving the vertex at $x+1$ to i and shifting the vertices in between to the right by one. We explicitly embed R^- onto $[i, j - |S|]$. This is possible because $B[i, j - |S|]$ is an independent set except for the single edge $\{i, i+1\}$ and R^- is not a central-star. Then if S is a central-star, we embed the center at $j - |S| + 1$, which is an isolated vertex in $[j - |S| + 1, j]$. If S is a dangling star, then we embed the root at j and the center at $j - |S| + 1$. Otherwise, S is not a star and we recursively embed S onto $[j - |S| + 1, j]$. Recall that $j - |S| + 1$ is a locally isolated vertex and $\{i, j - |S| + 1\} \notin E(B)$.

(because i is a leaf whose only neighbor is at $i + 1 \neq j - |S| + 1$). Therefore, there is no conflict for the recursive embedding and $B[j - |S| + 1, j]$ is not a star.

Case 2.3.3.2 The root of $B[i, x + 1]$ is at $x + 1$ (and possibly in conflict with r ; Figure 25a).

If S is a central-star, then we change the embedding of B as follows: First flip B^* so that its center is at i and then exchange the vertices at i (center c of B^*) and $i + 1$ (a leaf of B^*). Put r at i , which is a leaf of B^* and therefore not in conflict. Then put s at $x + 1$, whose only neighbor is (now) at x (originally at i), drawing the edge $\{r, s\}$ above the spine. Next put a leaf of S on $i + 1$ (center c of B^*), again drawing the edge $\{s, c\}$ above the spine. Put the remaining leaves of S on the vertices $[x + 2, j - 1]$, drawing the edges to s below the spine. This leaves us with a set of isolated vertices, all accessible from below the spine, on which we can complete an explicit embedding of R^- (Figure 25b).

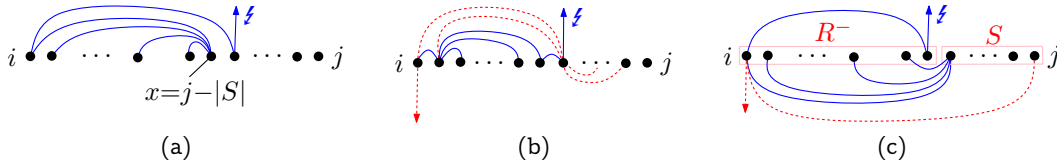


Figure 25: $\{i, x + 1\} \in E(B)$, $x = j - |S|$ and S is a central-star (Case 2.3.3.1).

A similar embedding also works for a dangling star S : Put r at $i + 2$ (which is another leaf of B^* because $|R^-| \geq 3$), put s at i and the center of S at $x + 1$.

Otherwise, S is not a star. Then we modify the embedding of B by drawing all edges of B^* below the spine and exchanging x and $x + 1$. Explicitly embed R^- onto $[i, j - |S|]$. This is possible because $B[i, j - |S|]$ is an independent set except for the edge $\{i, j - |S|\}$ and R^- is not a central-star. Recursively embed S onto $[j, j - |S| + 1]$ (Figure 25c). As $j - |S| + 1$ is a locally isolated vertex in $B[j - |S| + 1, j]$, we know that $B[j - |S| + 1, j]$ is not a star. There is no degree-conflict by assumption (Case 2.2 handles this scenario) and—as opposed to Case 2.3.1.2—we do not change $B\langle j \rangle$ here. Again by assumption $\{i, j\} \notin E(B)$ and so there is no edge-conflict for the recursive embedding of S , either. \square

10 Embedding the red tree: a large red star

In this section we handle the case where R^- is a star. If R^- is a star, then it must be a dangling star: otherwise, by the choice of S as a smallest subtree, R would be a star. Let q be the child of r in R^- and let $Q = t(q)$. Then Q is a central-star. Our default approach in this case is to explicitly embed Q and recursively embed $S^+ := R \setminus Q$. Note that $\deg_{S^+}(r) = 1$. Consequently, when we try to recursively embed S^+ onto some interval $[x, y]$, there can be a degree-conflict only if $B[x, y]$ is a star: a case we must handle separately anyway. Hence, for a recursive embedding of S^+ it suffices to check that we are not embedding against a star, to establish the placement invariant, and to check that there is no edge-conflict.

Proposition 19. *If R^- is a star, S^+ is not a star, and $\{i, j\} \notin E(B)$, then R and B admit an ordered plane packing onto $[i, j]$.*

Proof. Let $d := \deg_Q(q)$. We have $|S^+| \geq 4$ since S^+ is not a star. Hence, $|Q| \geq |S| \geq 3$. Flip $B\langle j \rangle$ if necessary to put its root at j . We first try the following. Use the red-star embedding to embed q at j and the children of q on the $\deg_Q(q)$ rightmost non-neighbors of j in $[i + 1, j - 1]$. Let I' be the interval of remaining vertices. Embed S^+ recursively onto I' . See Figure 26a.

Let us first consider the conditions under which the embedding of S^+ works. The embedding fails if $B[I']$ is a star, which happens only if (Case 1) $B^* := B\langle i \rangle$ is a star with $|B^*| \geq |S^+|$ and $\deg_B(j) = 0$. Otherwise, suppose there is an edge-conflict for embedding S^+ onto I' . Then

$B^* = B[I']\langle i \rangle$ is a central-star rooted at b^* . By choice of I' , we have $B^* = B\langle i \rangle$. By (I1), b^* has no edge to the parent of r . If it had an edge to j (which is where we embedded q), then by 1SR b^* is embedded at i . But then $\{i, j\} \in E(B)$, a contradiction. As argued at the start of this section, there can be no degree-conflict for S^+ . Hence, (I1) holds.

The embedding of Q works unless there is a degree-conflict for placing q onto j and embedding the children of q onto $[j-1, i+1]$, that is unless (Case 2) $\deg_Q(q) + \deg_B(j) \geq |I| - 1$. We deal with both cases below.

Case 1 $B^* := B\langle i \rangle$ is a star with $|B^*| \geq |S^+|$ and $\deg_B(j) = 0$. Let x be such that $B[i, x] = B^*$.

Case 1.1 $|B^*| > |S^+|$. Flip $B[i, x]$ if necessary to put its center at x . Embed q onto j and the children of q explicitly onto $[j-1, j - \deg_Q(q)]$. See Figure 26b. This works since $\deg_B(j) = 0$. Embed S^+ recursively onto $[j - \deg_Q(q) - 1, i]$. This works since $|B^*| > |S^+|$ and hence $B[i, j - \deg_Q(q) - 1]$ is an independent set and $j - \deg_Q(q) - 1$ is not the root of B^* .

Case 1.2 $|B^*| = |S^+|$. Flip $B[i, x]$ if necessary to put its center at i . By the peace invariant, this star-center is not in edge-conflict with r . Since $|S^+| \geq 4$, the blue vertex at $i+1$ is a leaf in B^* that is not the root of B^* . We change the blue embedding as follows. Simultaneously move $B[i+2, j]$ to $[i+1, j-1]$ and $i+1$ to j . The edge $\{i, j\}$ is drawn in the lower halfplane. Note that $B[i, x]\langle x \rangle$ is now an isolated vertex. Embed q onto $j-1$ and the children of q onto j and $[j-2, j - \deg_Q(q)]$. This works because $j-1$ is isolated. Embed r at i . Embed S recursively onto $[x, i+1]$ if S is not a star. See Figure 26c. Otherwise, S is a dangling star. In this case, embed s at x , the child s' of s onto $i+1$ and the children of s' onto $[i+2, x-1]$. This works because $j-1$ is isolated in B (and so $\{i, j-1\}$ is not used) and x is isolated in $B[i, x]$ (and so $\{i, x\}$ is not used and there is no conflict for embedding S onto $[x, i+1]$).

Case 2 $\deg_Q(q) + \deg_B(j) \geq |I| - 1$. Then $\deg_B(j) \geq |I| - 1 - \deg_Q(q) = |S^+|$ and so $|S^+| < |B\langle j \rangle|$. Let x and y such that $B[x, j] = B\langle j \rangle$ and $|[y, j]| = |S^+|$. Since $|S^+| < |B\langle j \rangle|$ we have $x < y$. Flip $B[x, j]$ to put its root at x . The proof of this case will not rely on the peace invariant, except in Case 2.3.3.

We first try the following. If x has a subtree that is not a central-star on at least $|S^+|$ vertices, then rearrange $B[x, j]$ to put a smallest such subtree at j . Embed q at $y-1$ and the children of q at $[y-2, i]$. Embed S^+ recursively at $[j, y]$. See Figure 26d. This fails immediately if (1) $B[y, j]$ is a star, in which case every subtree of x is a central-star on at least $|S^+|$ vertices or a dangling star on exactly $|S^+|$ vertices. Otherwise, suppose there is a edge-conflict for embedding S^+ . Then $B^* = B[j, y]\langle j \rangle$ is a star rooted at a center b^* . Since x is the only vertex on $[x, j]$ with edges to the outside of I , b^* must be adjacent to $y-1$ (which is where we placed q). By 1SR, we have $b^* = j$. Hence, we must handle the case (2) $\{y-1, j\} \in E(B)$ separately. This covers the possible issues with the recursive embedding of S^+ . The embedding of Q works unless (3) $y-1$ is not isolated in $B[i, y-1]$. We deal with these three cases next.

Case 2.1 $B[y, j]$ is a star. Then by the rearrangement of $B[x, j]$ performed above, every subtree of x is a central-star on at least $|S^+|$ vertices or a dangling star on exactly $|S^+|$ vertices.

Case 2.1.1 Some subtree U of x is a dangling star on exactly $|S^+|$ vertices. Re-embed $B[x, j]$,

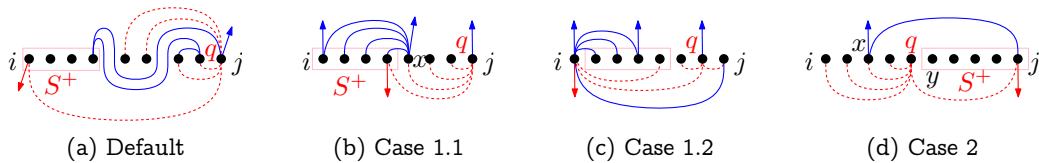


Figure 26: The case analysis in the proof of Proposition 19 (Part 1/6).

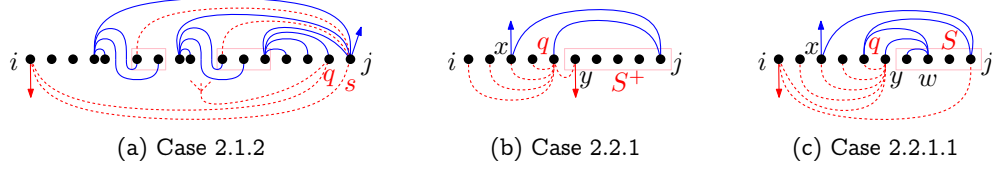


Figure 27: The case analysis in the proof of Proposition 19 (Part 2/6).

placing the root at j and U as the closest subtree. Afterwards, $U = B[y - 1, j - 1]$, the center of U is at $j - 1$, and j is isolated in $B[y, j]$. Embed r at i , S recursively onto $[j, y + 1]$, q onto y , and the children of q onto $[y - 1, i + 1]$. The embedding of S works because $\{i, j\} \notin E(B)$ and j is isolated in $B[y + 1, j]$. The embedding of Q works because y is a leaf of U and hence adjacent only to $j - 1$ in B .

Case 2.1.2 Every subtree of x is a central-star on at least $|S^+|$ vertices. Flip $B[x, j]$ to put its root at j . Embed r onto i and s onto j . We embed S explicitly, as follows. Let c_1, \dots, c_d be the children of S such that $t(c_1)$ is a largest subtree. Since S is not a central-star, $|t(c_1)| \geq 2$. Let v_1, \dots, v_k be the children of j ordered by proximity of j (v_1 is the closest). By the assumption of Case 2 we have $\deg_B j \geq |S^+|$ and hence $k \geq d + 1$. We embed $t(c_1)$ as follows. Reroute the edges of v_2 to its $|t(c_1)| - 1$ rightmost neighbors via the lower halfplane. Embed $t(c_1)$ explicitly on these $|t(c_1)| - 1$ rightmost neighbors of v_2 and on v_1 in the upper halfplane. For $i \geq 2$, we embed $t(c_i)$ as follows. Reroute the edges of v_{i+1} to its $|t(c_i)|$ rightmost neighbors via the lower halfplane and embed $t(c_i)$ explicitly on these vertices in the upper halfplane. Since we embedded a vertex of $t(c_1)$ on v_1 , $j - 1$ is isolated on the remainder. Embed q onto $j - 1$ and the children of q onto the remainder. See Figure 27a.

Case 2.2 $B[y, j]$ is not a star and $\{y - 1, j\} \in E(B)$. We consider two cases.

Case 2.2.1 $\{y - 1, y\} \notin E(B)$. Embed q onto $y - 1$ and the children of q onto $[y - 2, i]$. This works by 1SR and $\{y - 1, j\} \in E(B)$. Embed S^+ recursively onto $[y, j]$. See Figure 27b. Since $B[y, j]$ is not a star by assumption and since $B[x, j]$ is rooted at x , the only possible issue is an edge-conflict. In that case, let w such that $B[y, j] \setminus \{y\} = B[y, w]$. The root b^* of $B[y, w]$ is in edge-conflict with r . Due to the edge $\{x, j\}$ that is used by the blue embedding, the edge-conflict can be caused only by an edge from b^* to $y - 1$ (which is where we embedded q). By 1SR, $b^* = w$. Since $\{y - 1, y\} \notin E(B)$ and $B[y, j]$ is not a star, we have $y + 1 \leq w \leq j - 1$. We consider two cases.

Case 2.2.1.1 S is not a star. Embed r onto i , q onto y , the children of q onto $[y - 1, i + 1]$, and S recursively onto $[j, y + 1]$. See Figure 27c. Since $B[y, w]$ is a star and $y < w$, by 1SR y is isolated on $B[i, y]$: hence the embedding of Q works and the edges $\{r, q\}$ and $\{r, s\}$ incident to r are not used by the blue embedding. Hence, the only possible issues are caused by recursively embedding S . Suppose there is a conflict for embedding S onto $[j, y + 1]$. Then $B[j, y + 1] \setminus \{j\}$ is a central-star. Since $\{x, j\} \in E(B)$ and $x < y$ it follows that the root (and hence the center) of $B[j, y + 1] \setminus \{j\}$ is at j . But since $B[y - 1, w]$ is a subtree of j on more than one vertex, this violates

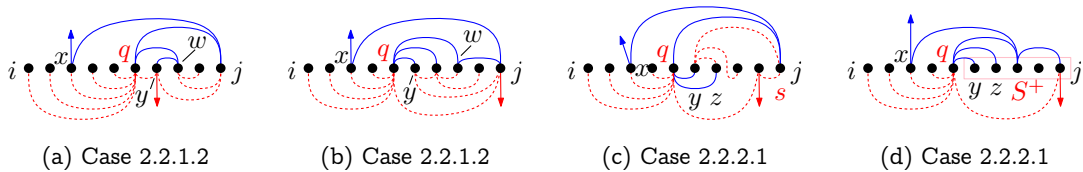


Figure 28: The case analysis in the proof of Proposition 19 (Part 3/6).

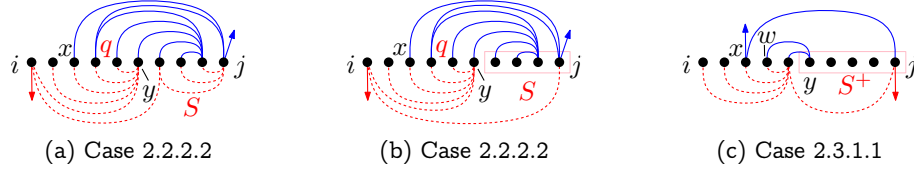


Figure 29: The case analysis in the proof of Proposition 19 (Part 4/6).

LSFR at j . Hence, there is no conflict for embedding S , which concludes this case.

Case 2.2.1.2 S is a star. Since S^+ is not a star, S is a dangling star. Let s' be the child of s . We distinguish two cases. If $w = y + 1$, then embed r onto y , q onto $y - 1$, the children of q onto $[y - 2, i]$, s onto j , s' onto $w = y + 1$, and the children of s' onto $[y + 2, j - 1]$. See Figure 28a. Since y is not the root of $B[i, x]$, y has no edges to the outside of the interval and hence it is safe to embed r there. Since $B[y - 1, w]$ is a star centered at w , y is adjacent only to w in the blue embedding and hence $\{r, s\} = \{y, j\} \notin E(B)$ and $\{r, q\} = \{y - 1, y\} \notin E(B)$. By 1SR, w is isolated in $B[w, j]$ and hence we can embed S as described and similarly y is isolated in $B[i, y]$ and hence we can embed Q as described.

If $w \geq y + 2$, then flip the blue embedding at $[y - 1, w]$. This places the center of the star $B[y - 1, w]$ at $y - 1$ and its root at w . Since $w \geq y + 2$, the vertices y and $y + 1$ are adjacent only to $y - 1$ now. Embed r onto j (which is not the root of $B[x, j]$), s onto y , s' onto $y + 1$, the children of s' onto $[y + 2, j - 1]$, q onto $y - 1$ (the edge $\{y - 1, j\}$ is no longer used after flipping), and the children of q onto $[y - 2, i]$. See Figure 28b. After flipping, $y - 1$ is isolated in $B[i, y - 1]$ and hence the embedding of Q works.

Case 2.2.2 $\{y - 1, y\} \in E(B)$.

Case 2.2.2.1 $B[y - 1, j]$ is not a star centered at $y - 1$. Let z be such that $B[y - 1, j - 1] \setminus \{y - 1\} = B[y - 1, z]$. Since $\{y - 1, y\} \in E(B)$ and $\{y - 1, j\} \in E(B)$ (Case 2.2), $B[y - 1, z]$ is a central-star. Since $B[y, j]$ is not a star (Case 2.2), $y \leq z \leq j - 2$. By LSFR at j we know that $\{j - 1, j\} \notin E(B)$. Since $z \leq j - 2$ we know that $\{y - 1, j - 1\} \notin E(B)$.

Suppose first that S is a dangling star and let s' be the child of s in S . Then embed r onto $j - 1$, q onto $y - 1$, the children of q onto $[y - 2, i]$, and s onto j . Flip $B[y - 1, z]$ into the lower halfplane. Embed s' onto y , drawing the edge $\{s, s'\}$ in the upper halfplane. Embed the children of s' onto $[y + 1, j - 2]$. The edges between s' and its children embedded at $[z + 1, j - 2]$ are drawn as biarcs. See Figure 28c.

Otherwise, S is not a star since S^+ is not a star. Flip $B[z + 1, j]$. Embed q onto $y - 1$ and the children of q onto $[y - 2, i]$. Embed S^+ recursively onto $[j, y]$. See Figure 28d. Since y is isolated in $B[y, j]$, $B[y, j]$ is not a star. If there is a conflict for the embedding of S^+ onto $[j, y]$, then $B[y, j] \setminus \{j\}$ must be a central-star rooted and centered at $z + 1$. But this violates the LSFR at j before flipping. Hence, this embedding works.

Case 2.2.2.2 $B[y - 1, j]$ is a star centered at $y - 1$. Let $B' := B[y - 1, j]$. We reembed $B[x, j]$ as follows. Use the normal embedding algorithm for blue trees to embed $B[x, j]$ onto $[j, x]$ (placing the root at j), but embed B' as the closest subtree, i.e., embed B' at $[y - 2, j - 1]$. This embeds the center of B' at $j - 1$.

Embed r onto i , q onto y , and the children of q onto $[y - 1, i + 1]$. This works so far: y is adjacent only to $j - 1$ in the blue embedding. If S is a star (it is not rooted at a center) then embed s onto $y + 1$, the child s' of s onto j , and the children of s' onto $[j - 1, y + 2]$. See Figure 29a. This works because j is isolated on $B[y - 1, j]$. If S is not a star, embed S recursively onto $[j, y + 1]$. See Figure 29b. Since j is isolated in $B[y + 1, j]$, $B[y + 1, j]$ is not a star and there is no conflict for embedding S onto $[j, y + 1]$.

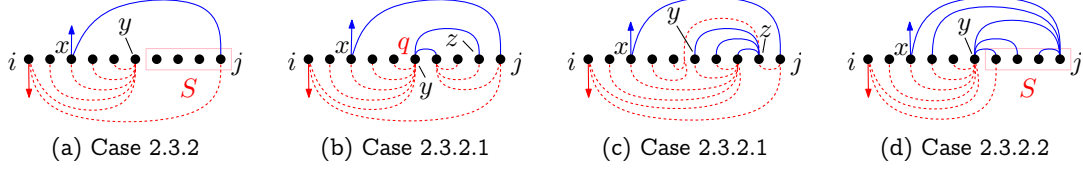


Figure 30: The case analysis in the proof of Proposition 19 (Part 5/6).

Case 2.3 $B[y, j]$ is not a star and $y - 1$ is not isolated in $B[i, y - 1]$. We distinguish three cases.

Case 2.3.1 $B[i, x]$ is not a star and y is not isolated in $B[i, y]$. By 1SR at $y - 1$, the edge $\{y - 1, y\}$ is not used. Let w be the rightmost neighbor of y on $[i, y - 1]$. Then $x \leq w$ and $B[w, y]$ is a tree on at least three vertices.

If $B[w, y]$ is a central-star, then its root and center is at w . Use the leaf-isolation-shuffle on $B[w, y]$ to place a leaf at $y - 1$ and its parent at y . By Proposition 8, if $B[w, y]$ is not a central-star, this places the root of $B[w, y]$ at w and preserves the 1SR at y . If $B[w, y]$ is a central-star, then let z be the largest index such that $B[w, z]$ is a central-star. Since $B[i, x]$ is not a star, we have $x < w$ and since $\{i, x\} \in E(B)$ we have $z \leq j - 1$. The leaf-isolation-shuffle places the root of $B[w, z]$ at y . Note that all vertices in $[y + 1, z]$ are now also adjacent to y .

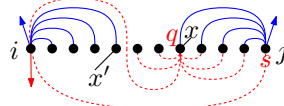
Case 2.3.1.1 $B[w, y]$ is not a central-star or $B[w, y]$ is a central-star but $z < j - 1$. In the latter case, the edge $\{w, j\}$ is not used, as this would imply that $\{w, z + 1\}$ is used by LSFR at w , contradicting the choice of z . Embed q onto $y - 1$ and the children of q onto $[y - 2, i]$. This works because $y - 1$ is adjacent only to y in B . Embed S^+ recursively onto $[j, y]$. See Figure 29c. Since the root of $B[x, j]$ is at x , any edge-conflicts must be caused by edges to $y - 1$ (which is where we embedded q). However, only y is adjacent to $y - 1$ in B and $B[y, j] \setminus \{y\} \neq B[y, j]$ by 1SR on y or by $z < j - 1$. Hence, there is no conflict for embedding S^+ onto $[j, y]$.

Case 2.3.1.2 $B[w, y]$ is a central-star with $z = j - 1$. Then $B[w, j]$ is a dangling star centered at w . Since $w < y$, we can proceed as in Case 2.2.2.2 (the argument still works for the larger star we have in this case).

Case 2.3.2 $B[i, x]$ is not a star and y is isolated in $B[i, y]$. Since $\{x, j\} \in E(B)$, it follows that y is not isolated in $B[y, j]$. We first try the following. Embed r onto i , q onto y , the children of q onto $[y - 1, i + 1]$, and S recursively onto $[j, y + 1]$. See Figure 30a. The embedding of Q works because y is isolated in $B[i, y]$. The embedding of S fails if (1) S is a star. In addition, the embedding could fail if $B[y + 1, j]$ is a star or if there is a conflict for embedding S onto $[j, y + 1]$, in which case $B[y + 1, j] \setminus \{y\}$ is a central-star. We cover these cases with (2) $B[y + 1, j]$ is a dangling star and (3) $[j, y + 1]$ is in conflict for embedding S .

Case 2.3.2.1 S is a star. Since S^+ is not a star, S is a dangling star centered at s' . Let z be such that $B[y, j] \setminus \{y\} = B[y, z]$. Suppose first that $B[y, z]$ is not a central-star. Use a leaf-isolation shuffle on $B[y, z]$ to put a leaf at $y + 1$, its parent at y , and the root at z . This works by Proposition 8. Embed r onto i , q onto y , and the children of q onto $[y - 1, i + 1]$. This works so far, since the leaf-isolation shuffle preserves the 1SR at y . Embed s onto j , s' onto $y + 1$, and the children of s' onto $[y + 2, j - 1]$. This works because $B \setminus \{i\} \neq B \setminus \{j\}$ and because $y + 1$ is isolated in $B[y + 1, j]$. See Figure 30b.

Otherwise, $B[y, z]$ is a central-star. Then it must be rooted and centered at z . By the assumption of Case 2.3, we must have $z < j$. Embed r onto i , s onto j , and s' onto z . This works so far since $B \setminus \{i\} \neq B \setminus \{j\}$ and $B[y, j] \setminus \{y\} \neq B[y, j] \setminus \{j\}$. Embed a child of s' on every vertex in $[z + 1, j]$. Exactly $|[y, z - 1]|$ children of s' remain to be embedded. Since $y - 1$ is not isolated in $B[i, y - 1]$ (assumption of Case 2.3), $y - 1$ and z must have some common parent p with $x \leq p \leq y - 2$. By LSFR at p , we have $|t(y - 1)| \geq |t(z)| > |[y, z - 1]|$, and so $t(y - 1)$ is large



(a) Case 2.3.3

Figure 31: The case analysis in the proof of Proposition 19 (Part 6/6).

enough to accomodate all remaining children of s' . This is true even if $x = p$ (for which we modified the order of the subtrees), since z is not the last subtree. Thus, embed the remaining children of s' onto $[y - 1, y - |y, z - 1|]$. Embed q onto the leaf $z - 1$ of z and the children of q onto the remainder. See Figure 30c.

Case 2.3.2.2 $B[y + 1, j]$ is a dangling star. Since y is not isolated in $B[y, j]$, we must have $\{y, j\} \in E(B)$. Simultaneously shift $B[y + 1, j - 1]$ to $[y, j - 2]$ and y to $j - 1$. Embed r onto i , q onto y , the children of q onto $[y - 1, i + 1]$, and S recursively onto $[y + 1, j]$. Since $y + 1$ is isolated in $B[y + 1, j]$, the recursive embedding of S always works. See Figure 30d.

Case 2.3.2.3 $[j, y + 1]$ is in conflict for embedding S . Let w be such that $B[y + 1, j]\langle j \rangle = B[w, j]$. Then $B[w, j]$ is a central-star rooted at j . If $w = y + 1$, then y must be connected to j and so $B[y, j]$ is a star. This contradicts our assumption of Case 2.3 and hence $w \geq y + 2$. The root j of $B[w, j]$ cannot be in edge-conflict with s because $B\langle i \rangle \neq B\langle j \rangle$. Thus, it is in degree-conflict and we have $\deg_B(j) + \deg_S(s) \geq |y, j|$. Recall from the start of Case 2 that the root of $B[x, j]$ has degree at least $|S^+|$. Since $B[w, j]$ is not an isolated vertex, all other subtrees of x must have size at least two. Thus, $|[x + 1, w - 1]| \geq 2(|S^+| - 1) \geq |S|$. Embed r onto i . Use a blue-star embedding to embed S onto $[j, x + 1]$. Note that $B[x + 1, j]\langle j \rangle = B[w, j]$ and that (BS2) is satisfied by the discussion above. Embed q onto y and the children of q onto $[y - 1, i + 1]$ to complete the embedding.

Case 2.3.3 $B[x, j]$ is a star. Recall that $B[x, j]$ is rooted at x . Since $\deg_B(j) \geq |S^+| \geq 4 > 1$, $B[x, j]$ is a central-star. Then the rearrangement of $B[x, j]$ at the start of Case 2 did not change anything, and hence B satisfies the invariants. We replay the case analysis, starting from the very start of this proof, but now we embed on $[i', j'] := [j, i]$ (i.e. we embed from the other side). Note that $[i', j']$ may not satisfy the peace invariant, but it satisfies the other invariants. Consider the initial embedding in the proof, which performs a red-star embedding of Q from j' and then embed S^+ on the left of $[i', j']$. The embedding of S^+ always works: $B[x, j]$ is a star of size larger than $|S^+|$ which appears on the left of the interval $[i', j']$, and hence the first $|S^+|$ elements of $B[i', j']$ form an independent set. Thus, if the initial embedding fails, we must land in Case 2. Since we have not yet used the peace invariant in Case 2 so far, we can simply execute the case analysis of Case 2 until we get an embedding or we arrive at this case (Case 2.3.3).

It remains to consider the event that the embedding procedure also reaches this case (Case 2.3.3) for embedding R onto $[j', i']$. Refer to Figure 31a. Then $B\langle i \rangle$ and $B\langle j \rangle$ are both central-stars of size larger than $|S^+|$. Flip $B\langle i \rangle$ if necessary to put its root at i and flip $B\langle j \rangle$ if necessary to put its root at j . Let x' be such that $B[i, x'] = B\langle i \rangle$. By the peace invariant for embedding R on $[i, j]$, the root of $B\langle i \rangle$ at i is not in edge-conflict with r . Embed r at i and q at x , drawing the edge $\{r, q\}$ as a biarc that is in the upper half-plane near r and crosses the spine between x' and $x' + 1$. Embed a child of q on every vertex in $[x' + 1, x - 1] \cup [x + 1, j - 1]$. Using that $|B\langle i \rangle| \geq |S^+| + 1$, this works because $\deg_Q(q) = |I| - |S^+| - 1 \geq |I| - |B\langle i \rangle| = |[x' + 1, j]| > |[x' + 1, x - 1] \cup [x + 1, j - 1]|$. Embed s onto j . The remaining blue vertices in $[i + 1, x']$ form an independent set on which we can easily embed the remaining children of s q and S explicitly.

□

Proposition 20. *If R^- and S^+ are both stars and $\{i, j\} \notin E(B)$, then R and B admit an ordered plane packing onto $[i, j]$.*

Proof. Let q be the child of r in R^- and let $Q = t(q)$. Then Q is a star centered at q and S is a star centered at s . The case $|S| = 1$ is handled by Lemma 16. In the remainder we assume $|S| \geq 2$. We deal with two red stars here, so we frequently use the red-star embedding. Since all embeddings in this proof are explicit (we cannot recursively embed stars, after all), we only perform Step 1 (Embed) of the red-star embedding for ease of explanation.

Let h such that $B[h, j] = B\langle j \rangle$. Re-embed $B[h, j]$ by putting its root at j and embedding its subtrees according to the smaller-subtree-first rule (SSFR) and the 1SR. By assumption, $\{i, j\} \notin E(B)$ and hence these modifications do not touch $B\langle i \rangle$. Our general plan is the following. Embed r at i . This works by the placement invariant. Perform a red-star embedding to embed s onto j and the children of s onto the rightmost $\deg_S(s)$ non-neighbors of j in $[i + 1, j - 1]$. Since $\{i, j\} \notin E(B)$, j is not in edge-conflict with s and hence (RS1) holds. Hence, this works unless (RS2) fails, i.e., unless (1) $\deg_S(s) + \deg_B(j) \geq |[i + 1, j - 1]| + 1 = |I| - 1$. We embed q onto the rightmost child h' of j . This works unless j has no children, i.e., unless (2) $\deg_B(j) = 0$. We finally embed the children of q onto the remaining vertices. See Figure 32a. Since the red-star embedding ensures that all remaining vertices are visible from below, this is possible unless h' has an edge to a remaining vertex. Note that all edges of h' are in $B[h', j]$, and we embedded s onto j . Hence, it suffices to handle the case where the red-star embedding did not embed a child of s onto every vertex of $B[h' + 1, j - 1]$, i.e., the case that (3) $\deg_S(s) \leq |[h' + 1, j - 1]| - 1$. We deal with these remaining cases below. We first state a useful observation.

Observation 21. *Let $b, b' \in B$ be the roots of two different trees of the forest B . Suppose that $\deg_B(b) + \deg_S(s) \geq |I| - 1$. Then*

(P1) $\deg_B(b) \geq (|I| + 1)/2$;

(P2) *at least three children of b are leaves; and*

(P3) $\deg_B(b') + \deg_S(s) \leq \deg_B(b') + \deg_Q(q) \leq |I| - 3$

Proof. Since r has two subtrees and S is the smaller one, we have $|S| \leq (|I| - 1)/2$ and hence $\deg_S(s) = |S| - 1 \leq (|I| - 3)/2$. By the assumption, we have $\deg_B(b) \geq |I| - 1 - \deg_S(s) \geq |I| - 1 - (|I| - 3)/2 = (|I| + 1)/2$, as claimed in (P1). Let λ be the number of leaf subtrees of b . The other subtrees of b have size at least two and the total size of $B\langle b \rangle$ is at most $|I| - 1$, since b and b' are the roots of different trees. Hence, $1 + \lambda + 2(\deg_B(b) - \lambda) \leq |B\langle b \rangle| \leq |I| - 1$, and so $\lambda \geq 1 + 2\deg_B(b) - (|I| - 1) = 2 - |I| + 2\deg_B(b)$. Then, by (P1), $\lambda \geq 2 - |I| + (|I| + 1) = 3$, which proves (P2).

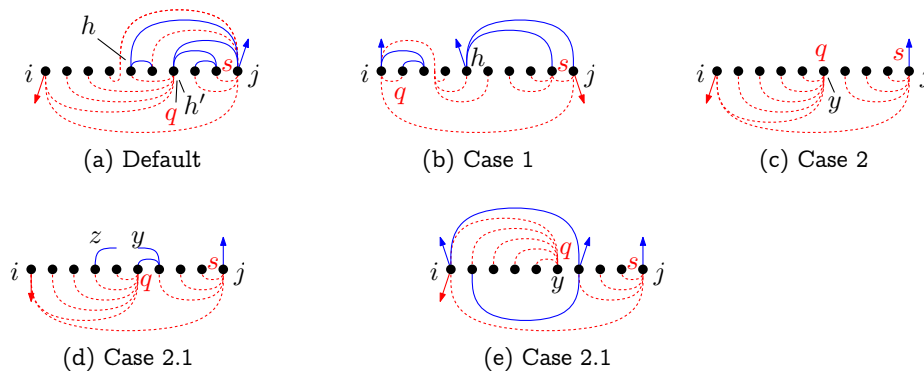


Figure 32: The case analysis in the proof of Proposition 20 (Part 1/4).

Since r has two subtrees and Q is the larger one, we have $|Q| \geq (|I| - 1)/2$ and hence $\deg_Q(q) = |Q| - 1 \geq (|I| - 3)/2$. The first inequality of (P3) follows from $|S| \leq |Q|$. Suppose towards a contradiction that the second inequality of (P3) is false, that is, $\deg_B(b') + \deg_Q(q) \geq |I| - 2$. Adding this equation to the assumption, we obtain $\deg_B(b) + \deg_B(b') + \deg_S(s) + \deg_Q(q) \geq 2|I| - 3$. Since $\deg_S(s) + \deg_Q(q) = |I| - 3$, it follows that $\deg_B(b) + \deg_B(b') \geq |I|$, which contradicts $b \neq b'$. Claim (P3) follows. \square

Case 1 $\deg_S(s) + \deg_B(j) \geq |I| - 1$. Then Observation 21 applies with $b := j$. Re-embed $B[h, j]$ by placing its root at h and embedding its subtrees with LSFR and 1SR. Embed r onto j and s onto $j - 1$. This works because j and $j - 1$ are leaves by (P2) and LSFR. If necessary, flip $B\langle i \rangle$ to put its root at i . Use the red-star embedding to embed q onto i and the children of q onto the leftmost $\deg_Q(q)$ non-neighbors of i in $[i + 1, j - 2]$. (RS1) holds since $\{i, j\} \notin E(B)$. (RS2) holds since $|[i + 1, j - 2]| = |I| - 3$ and by (P3) with $b := j$ and $b' := i$. Let x be the largest index on which a child of q was embedded. Then $|[i, x]| \geq 1 + \deg_Q(q)$. Since $\deg_B(j) \geq (|I| + 1)/2$ by (P1) we have $|[h, j]| \geq (|I| + 3)/2$. Then $|[i, x]| + |[h, j]| \geq 1 + (|I| - 3)/2 + (|I| + 3)/2 = |I| + 1$. It follows that $x \geq h$, and so the red-star embedding embedded a child of q onto h . Since this is the only vertex in B adjacent to $j - 1$ (which is where we embedded s), we can embed the children of s on the remainder. See Figure 32b.

Case 2 $\deg_B(j) = 0$. Let y such that $|[i, y]| = 1 + |Q|$. If y is isolated in $B[i, y]$ then embed r onto i , q onto y , the children of q onto $[y - 1, i + 1]$, s onto j , and the children of s onto $[j - 1, y + 1]$. See Figure 32c. This works due to the placement invariant and the fact that y is isolated in $B[i, y]$ and j is isolated in B . Otherwise, y is not isolated in $B[i, y]$. We distinguish two cases.

Case 2.1 $y + 1$ is not isolated in $B[i, y + 1]$. Let z be the rightmost neighbor of $y + 1$ in $B[i, y + 1]$. We have $i \leq z \leq y$. If $i < z$, then perform a leaf-isolation-shuffle on $B[z, y + 1]$ to put a leaf at y and its parent at $y + 1$. Embed r onto i . Since $i < z$, the blue vertex at i was not changed and hence this works by the placement invariant. Embed q onto y and the children of q onto $[i + 1, y - 1]$. This works since y is adjacent only to $y + 1$ in B . Finally, embed s onto j and the children of s onto $[j - 1, y + 1]$. This works because j is isolated in B . See Figure 32d.

Otherwise, $i = z$. Since z was chosen as the rightmost vertex of $y + 1$ in $B[i, y + 1]$, we have $\deg_{B[i, y + 1]}(y + 1) = 1$ and hence $\{i, y\} \in E(B)$. Flip $B[i, y + 1]$. After flipping, $\{i, i + 1\} \notin E(B)$. Embed r onto $i + 1$ and q onto i . Flip $B[i + 1, y + 1]$ into the lower halfplane and embed the children of q onto $[i + 2, y]$. This works because after flipping, i is adjacent only to $y + 1$ in $B[i, y + 1]$. Finally, embed s onto j and the children of s onto $[j - 1, y + 1]$. This works because j is isolated in B . See Figure 32e.

Case 2.2 $y + 1$ is isolated in $B[i, y + 1]$. In other words, all (possibly zero) edges incident to $y + 1$ leave $y + 1$ to the right. We distinguish two cases.

Case 2.2.1 $B\langle i \rangle$ is a central-star. If $B\langle i \rangle = B\langle y \rangle$ then use Lemma 17 to compute an ordered plane packing. Otherwise $B\langle i \rangle \neq B\langle y \rangle$. Flip $B\langle i \rangle$ if necessary to put its root at i .

If $|B\langle i \rangle| = 1$ then flip $B\langle y \rangle$ if necessary to put the root away from y (recall that y is not isolated in $B[i, y]$). Embed r onto y , q onto i , the children of q onto $[i + 1, y - 1]$, s onto j and the children of s onto $[j - 1, y + 1]$. See Figure 33a. This works because i and j are both isolated in B .

If $|B\langle i \rangle| \geq 2$, then we change the blue embedding as follows. Simultaneously shift $B[i + 2, j]$ to $[i + 1, j - 1]$ and $i + 1$ to j . The new edge $\{i, j\}$ is drawn in the lower halfplane. Afterwards, y is isolated in $B[i, y]$ and $j - 1$ is isolated in B . Embed r onto i . By the peace invariant, i is not in edge-conflict with r . Embed q onto y and the children of q onto $[y - 1, i + 1]$. Embed s onto $j - 1$ and the children of s onto j and $[j - 2, y + 1]$. See Figure 33b.

Case 2.2.2 $B\langle i \rangle$ is not a central-star. Then $|B\langle i \rangle| \geq 3$. Flip $B\langle i \rangle$ if necessary to put its root at i . Let z such that $B[i, z] = B\langle i \rangle$ and let x be the leftmost neighbor of i . Then $i < i + 2 \leq x \leq z$.

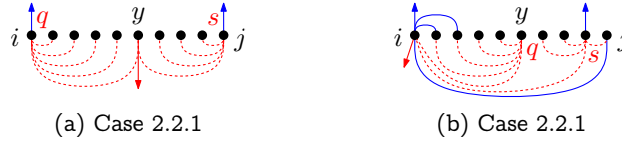


Figure 33: The case analysis in the proof of Proposition 20 (Part 2/4).

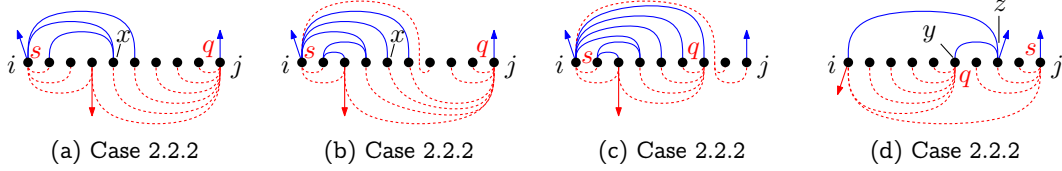


Figure 34: The case analysis in the proof of Proposition 20 (Part 3/4).

If $\deg_S(s) \leq |[i+1, x-2]|$ then embed r onto $i+1 + \deg_S(s)$, s onto i , the children of s onto $[i+1, i + \deg_S(s)]$, q onto j , and the children of q onto $[j-1, i+2 + \deg_S(s)]$. See Figure 34a.

Otherwise, $\deg_S(s) \geq |[i+1, x-2]| + 1$. Embed r onto $x-1$ and s onto i . Embed children of s onto $[i+1, x-2]$. Use the red-star embedding to embed the remaining children of s onto the $\deg_S(s) - |[i+1, x-2]|$ leftmost non-neighbors of i in $[x+1, j-1]$. If (RS2) is not violated, we complete the embedding by placing q at j and embedding the children of q on the remainder. See Figure 34b. If (RS2) is violated, then $\deg_S(s) - |[i+1, x-2]| + \deg_{B[x+1, j]}(i) > |[x+1, j-1]|$. Equivalently, $\deg_S(s) + \deg_B(i) \geq |I| - 2$. It follows that $\deg_B(i) \geq |I| - 2 - \deg_S(s) \geq |I| - 2 - (|I| - 3)/2 = (|I| - 1)/2$. Since $|Q| \geq |S| \geq 2$ we have $|I| \geq 5$. Hence $\deg_B(i) \geq (5 - 1)/2 = 2$. Instead of performing the red-star embedding on $[x+1, j-1]$, we now perform it on $[x+1, j]$. If (RS2) is not violated, then since our first red-star embedding failed, the remaining vertices are exactly the neighbors of i in B , which form an independent set. Complete the embedding by placing q at the rightmost neighbor of i (which is not adjacent to r) and the children of q on the remainder. See Figure 34c.

It remains to consider the case where (RS2) is again violated. In this case we have $\deg_S(s) + \deg_B(i) \geq |I| - 1$. Due to the degree-conflict and the fact that $\deg_S(s) = |[y+2, j]|$ we have $z \geq y+2$. Flip $B[i, z]$ to put its root at z . Observation 21 applies with $b := z$. By LSFR and (P2), i is a leaf of $B[i, z]$. We want to apply Observation 14 on $B[i+1, z]$. We first argue that the preconditions are satisfied. Let $n := |B[i+1, z]| - 1$ and $t := \deg_B(z) - 1$. Then $n \leq |[i+1, j-1]| - 1 \leq |I| - 3$ and by (P1) $t \geq (|I| + 1)/2 - 1 = (|I| - 3)/2 + 1 \geq n/2 + 1$, as required. Apply Observation 14 with $k := |[i+1, y-1]|$ and rearrange $B[i+1, z]$ to put the corresponding subtrees at $[i+1, y-1]$. Afterwards, all edges adjacent to y leave y to the right. Embed r onto i , q onto y , the children of q onto $[y-1, i+1]$, s onto j , and the children of s onto $[j-1, y+1]$. See Figure 34d. This works since y is isolated in $B[i, y]$ and j is isolated in B .

Case 3 $\deg_S(s) \leq |[h'+1, j-1]| - 1$. Recall that we re-embedded $B[h, j]$ by placing the root at j and embedding the subtrees of j according to the SSFR and 1SR. We defined h' as the rightmost child of j . We have $|t(h')| = |[h', j-1]| \geq 2 + \deg_S(s) \geq 3$. Hence, all subtrees of j have size at least $1 + |S| \geq 3$. We distinguish two cases.

Case 3.1 $\deg_B(j) \geq 2$.

Case 3.1.1 All subtrees of j are central-stars. Then we flip $B[h, j]$, placing its root at h . Embed r onto i , s onto j , and the children of s onto the rightmost $\deg_S(s)$ non-neighbors of j in $[h+1, j-1]$. Each edge is drawn with a biarc that is in the upper halfplane close to j . This works because $\deg_B(h) \geq 2$ (by our assumption and after flipping $B[h, j]$) and since all subtrees of h have size

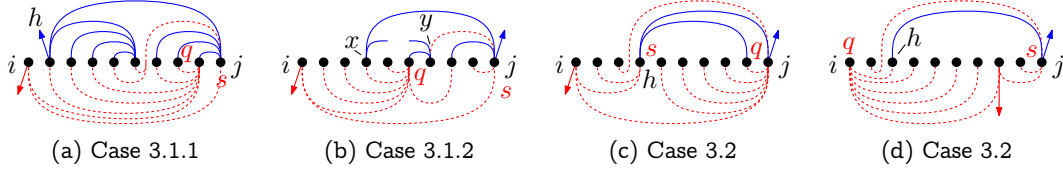


Figure 35: The case analysis in the proof of Proposition 20 (Part 4/4).

at least $1 + |S|$. Since $j - 1$ is adjacent only to j (which is where we embedded s), we can safely place q on $j - 1$ and the children of q on the remainder. See Figure 35a.

Case 3.1.2 Some subtree Z of j is not a central-star. Re-embed $B[h, j]$, putting the root at j and embedding the subtrees of j in any order that places Z leftmost. Let x and y with $x < y$ such that $Z = B[x, y]$. By Proposition 8 and since Z is not a central-star, we can use the leaf-isolation shuffle to place a leaf of $B[x, y]$ at $y - 1$, its parent at y , and the root at x . Embed r onto i . This works because of the placement invariant. Embed q onto $y - 1$ and s onto j . This works since $y - 1$ is incident only to y in B and $\{i, j\} \notin E(B)$. Embed a child of s onto y , drawing the edge in the upper halfplane. This works because $\deg_B(j) \geq 2$ and Z is the leftmost subtree of j . Embed the remaining children of s onto the rightmost vertices of $[i, j - 1]$. This works because all subtrees of j have size at least $1 + |S|$. Finally, note that we already embedded a vertex on the only blue vertex incident to $y - 1$, and hence we can embed the children of q onto the remainder. See Figure 35b.

Case 3.2 $\deg_B(j) = 1$. We first try the following. Embed r onto i . Use the red-star embedding to embed q onto j and the children of q onto the rightmost $\deg_Q(q)$ non-neighbors of j in $[i + 1, j - 1]$. (RS1) holds since $\{i, j\} \notin E(B)$. Since $|S| \geq 2$ we have $\deg_Q(q) \leq |I| - 4$ and hence $\deg_Q(q) + \deg_B(j) \leq |I| - 3 < |[i + 1, j - 1]|$. This establishes (RS2). Embed s onto h and the children of s onto the remainder. See Figure 35c. This works unless some remaining vertex is adjacent to h .

Since all neighbors of h are in $[h + 1, j]$, this implies $|Q| \leq |[h + 1, j]| - 1$, or equivalently, $|[h, j]| \geq |Q| + 2$. Embed r onto $h + \deg_Q(q)$. By $|[h, j]| \geq |Q| + 2$, this is not j and hence there is no edge-conflict. Embed q onto i and s onto j . This works because all neighbors of $h + \deg_Q(q)$ (which is where we placed r) in the blue embedding are in $[h, j - 1]$ since $\deg_B(j) = 1$. Embed the children of q onto $[h, h + \deg_Q(q) - 1]$. This works because $B\langle i \rangle \neq B\langle j \rangle$. Embed the children of s onto $[j - 1, h + \deg_Q(q) + 1]$ and $[h - 1, i + 1]$ (with biarcs). This works because the only neighbor of j is at h . See Figure 35d. \square

Proposition 22. *If R^- is a star and $\{i, j\} \in E(B)$, then R and B admit an ordered plane packing onto $[i, j]$.*

Proof. The presence of edge $\{i, j\} \in E(B)$ implies that B is a tree. In this case, we discard the initial embedding of B . Instead, we embed R using Algorithm 1, and then *re-embed* B using the additional information that R^- is a star.

To simplify notation, we exchange the roles of R and B . Refer to Figure 36a–36b. That is, we assume that B has been embedded using Algorithm 1, its root is at j , and it is composed of two trees S_B and B^- (corresponding to S and R^- , respectively): $S_B = B[i, x]$ is a tree of size $|S_B| \geq 2$ rooted at i , and $B^- = B[x + 1, j]$ is a star of size $|B^-| \geq (|I| + 1)/2$ centered at $x + 1$ and rooted at j . We do not make any assumption about S_B , apart from that it fulfills invariants (I1) and (I2). It remains to embed R onto $[i, j]$. Let S be a smallest subtree of R and let $R^- = R \setminus S$.

If $\deg_R(r) = 1$, then Lemma 15 completes the proof. If $|S| = 1$, then Lemma 16 completes the proof. Hence we may assume $\deg_R(r) \geq 2$ and $|S| \geq 2$. Since S is a smallest of two or more subtrees of r , we have $|S| \leq (|I| - 1)/2$ and $\deg_{R^-}(r) \leq (|R^-| - 1)/2$. Let z be such that

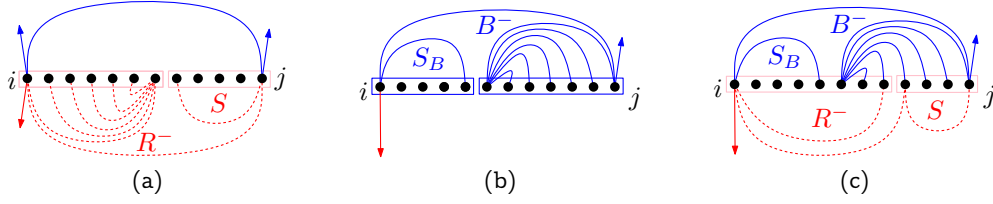


Figure 36: When the default embedding of both R and B contains edge $\{i, j\}$, we exchange the roles of R and B to simplify notation.

$|R^-| = |B[i, z]|$. Since $|B^-| \geq (|I| - 1)/2$, it follows that $z + 1$ is a leaf of the star B^- , and $B[z + 1, j]$ consists of isolated vertices.

Our first option to embed R is the following. Embed S onto $[z + 1, j]$ using Algorithm 1, and then embed R^- recursively onto $[i, z]$; see Figure 36c. This works unless $[i, z]$ is in degree-conflict with R^- , or R^- is a star. We consider these two possibilities separately.

Case 1 $[i, z]$ is in degree-conflict with R^- , but R^- is not a star. In this case, $S_B = B[i, z]\langle i \rangle$ is a central-star rooted at i and $\deg_{B[i, z]}(i) + \deg_{R^-}(r) \geq |R^-|$. Note that $B[i, j - 1]$ consists of two central-stars, $S_B = B[i, x]$ (rooted at i) and $B[x + 1, j - 1]$ (rooted at $x + 1$). Since $|R^-| \geq (|I| + 1)/2$ and $|S_B| \leq (|I| - 1)/2$ we have $z \geq x + 1$.

Case 1.1 $z \geq x + 2$. Embed S explicitly onto $[z + 1, j]$ and R^- recursively onto $[z, i]$. Since $z \geq x + 2$, the blue vertex at z is a leaf that is adjacent only to $x + 1$. Hence, there is no edge-conflict for the recursive embedding of R^- . Since $B[i, z]\langle z \rangle$ is a central-star, there could be a degree-conflict. In this case we have $\deg_{B[i, z]}(x + 1) + \deg_{R^-}(r) \geq |R^-|$. Adding this equation to the equation for the degree-conflict at $[i, z]$, we obtain $2|R^-| - 2\deg_{R^-}(r) \leq \deg_{B[i, z]}(i) + \deg_{B[i, z]}(x + 1) = |R^-| - 2$. It follows that $\deg_{R^-}(r) \geq |R^-|/2 + 1$. Since each subtree of r in R^- has size at least $|S| \geq 2$, we get $\deg_{R^-}(r) \geq (1 + 2\deg_{R^-}(r))/2 + 1 > \deg_{R^-}(r) + 1$, a contradiction. Hence, there is no degree-conflict and the recursive embedding of R^- always works.

Case 1.2 $z = x + 1$. In this case $|R^-| = (|I| + 1)/2$ and $|S| = (|I| - 1)/2$. Hence, r is binary in R . Let $Q = t(q)$ be the subtree of r in R^- . Embed r onto i , Q explicitly onto the independent set at $[z, i + 1]$, and S explicitly onto the independent set at $[z + 1, j]$. Since the blue embedding uses neither $\{i, z\}$ nor $\{i, z + 1\}$, this always works.

Case 2 R^- is a star. Since $|S| \geq 2$ and S is a smallest subtree of r , R^- is a dangling star, that is, it is centered at the unique child q of r in R^- . In this case, R and B have even more similarities: their roots each have two children, and both B^- and R^- are dangling stars. See Figure 37a.

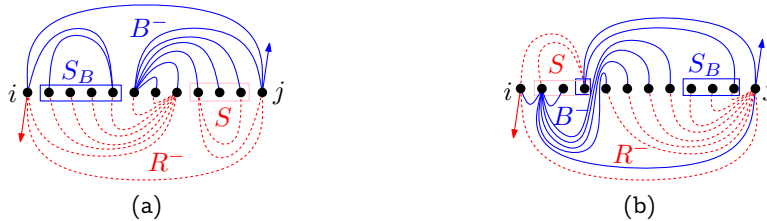


Figure 37: (a) In Case 2, the default embeddings of B and R share several edges. (b) We embed R and B explicitly (right).

We embed B and R simultaneously such that the root of S_B and S are mapped to the same point, and all other vertices of S_B and S are disjoint. This is possible since S_B and S each have size at most $(|I| - 1)/2$. Refer to Figure 37b. Embed the star B^- on $\{j\} \cup [i, j - |S_B|] \setminus \{i + |S|\}$ such that its root (which is the root of B) is embedded at j and its center at $i + 1$. The edges between

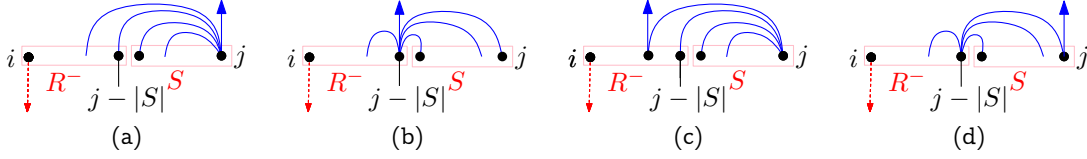


Figure 38: Moving the center of star B^{**} from j to $j - |S|$: when B^{**} is rooted at its center (a–b), and when it is rooted at a leaf (c–d).

the center $i + 1$ and other vertices of $[i, i + |S| - 1]$ are semicircles *below* the spine; the edge $\{i + 1, j\}$ is also a semicircle below the spine; and the edges between $i + 1$ and $[i + |S| + 1, j - |S_B|]$ are biarcs that start from $i + 1$ below the spine and cross the spine right after $i + |S|$. Embed the subtree S_B onto $\{i + |S|\} \cup [j - |S_B| + 1, j - 1]$ using Algorithm 1, with semicircles above the spine. Embed the tree R^- on $\{i\} \cup [i + |S|, j]$ such that its root (which is the root of R) is at i , and its center is at j , using semicircles below the spine. If S is not a star, then finish by embedding S onto $[i + |S|, i + 1]$ recursively, such that the edge $\{r, s\}$ and all edges of S are semicircles *above* the spine. If S is a central-star, embed S explicitly onto $[i + |S|, i + 1]$ above the spine. If S is a dangling star, then $|S| \geq 3$. Flip the blue embedding at $[i + 1, i + |S| - 1]$, placing the star-center of B^- at $i + |S| - 1$. Since $|S| \geq 3$, $\{i, i + 1\} \notin E(B)$. Embed s onto $i + 1$, the child s' of s onto $i + |S|$, and the children of s' onto $[i + |S| - 1, i + 2]$ (all above the spine).

It remains to show that we can recursively embed S as described above when S is not a star. Note that $B[i + 1, i + |S|]$ consists of an isolated vertex at $i + |S|$ (the root of S_B), and a star $B[i + 1, i + |S| - 1]$ centered and rooted at $i + 1$. Hence (I1) and (I2) follow. \square

Proposition 19, Proposition 20, and Proposition 22 together prove the following.

Lemma 23. *If R^- is a star, then R and B admit an ordered plane packing onto $[i, j]$.*

11 Embedding the red tree: a small blue star

In this section, we consider the case that $B[j - |S| + 1, j]$ is a star, but $B[i, j - |S|]$ is not a star. The size of the star is $|S|$. Due to Lemma 16, we may assume $|S| \geq 2$. Note, however, that $B[j - |S| + 1, j]$ may be part of a larger star within B . Let B^{**} be the maximal star in B that contains $B[j - |S| + 1, j]$. Note that the tree $B\langle j \rangle$ may be larger than B^{**} . Clearly, we have $|S| = |B[j - |S| + 1, j]| \leq |B^{**}|$. Due to 1SR, the center and the root of B^{**} are each located at either j or the leftmost vertex of B^{**} , which may be outside of the interval $[j - |S| + 1, j]$. We distinguish two cases: either $|S| < |B^{**}|$ (Proposition 24) or $|S| = |B^{**}|$ (Proposition 25, Proposition 26, and Proposition 27). These cases are tackled below.

Proposition 24. *If $B[j - |S| + 1, j]$ is a star and $2 \leq |S| < |B^{**}|$, then R and B admit an ordered plane packing onto $[i, j]$.*

Proof. By Lemma 17, we may assume that $B[i, j - |S|]$ is not a star. By Lemma 23, we may assume that R^- is not a star. Recall that the center of B^{**} is j , and its root is either j or the leftmost vertex of B^{**} . We start by rearranging the tree B^{**} such that its center moves to $j - |S|$; see Figure 38. If B^{**} is rooted at its center, then the root automatically moves to $j - |S|$, as well. Otherwise B^{**} is rooted at a leaf, which is the leftmost vertex of B^{**} and the root of the entire tree $B\langle j \rangle$ due to 1SR, and then we move the root of B^{**} to j . In both cases, $B[j - |S| + 1, j]$ consists of $|S|$ isolated vertices, and $B[i, j - |S|]$ continues to fulfill invariant (I2).

Case 1 $B[i, j - |S|]\langle i \rangle$ is not a central-star of size at least $|R^-| - \deg_{R^-}(r) + 1$. In this case, we embed S explicitly onto $[j - |S| + 1, j]$ and then R^- recursively onto $[i, j - |S|]$. Since $B[j - |S| + 1, j]$ consists of isolated vertices and $j - |S| + 1$ is not adjacent to i , the embedding of S always

works. We can embed R^- on $[i, j - |S|]$ because it fulfills invariants (I1) and (I2). Invariant (I2) holds by construction. If $B[i, j - |S| \langle i \rangle]$ is not a central-star, then (I1) is immediate; otherwise $B[i, j - |S| \langle i \rangle]$ is a central-star of size at most $|R^-| - \deg_{R^-}(r)$. Hence, r has no degree-conflict with $[i, j - |S|]$, and (I1) follows.

Case 2 $B[i, j - |S| \langle i \rangle]$ is a central-star of size at least $|R^-| - \deg_{R^-}(r) + 1$. Let $B^* := B[i, j - |S| \langle i \rangle]$. We claim that $B \langle i \rangle \neq B \langle j \rangle$. Suppose that $B \langle i \rangle = B \langle j \rangle$ for the sake of contradiction. Then before rearranging B^{**} , we had $\{i, j\} \in E(B)$. By LSFR and since B is not a star, the root of B was not at i . Again by LSFR, the root could have been at j only if B^{**} is a dangling star. But then, since $|B^{**}| > |S|$, $B[j - |S| + 1, j]$ was not a star to begin with: a contradiction. The claim follows. Since $|B \langle j \rangle| \geq |B^{**}| > |S|$, we have $B[i, j - |S| \langle i \rangle] = B \langle i \rangle$ and so $B^* = B \langle i \rangle$.

Since R^- has a degree-conflict with $[i, j - |S|]$, we follow a different strategy. We first blue-star embed R^- from $\sigma = i$ with $\varphi = (i + |B^*| + 1, \dots, i + |B^*| + d)$, and then embed S on $[j - |S| + 1, j]$. The conditions for the blue-star embedding are met: (BS1) holds by (I3) for embedding R onto $[i, j]$; for (BS2) on the one hand $|R^-| \leq |B^*| + \deg_{R^-}(r) - 1 \leq |B^*| + \deg_{R^-}(r)$ and on the other hand, by (I1), we have $|B^*| \leq |R| - \deg_R(r)$ and so $|B^*| + \deg_{R^-}(r) \leq |R| - 1$. As $B^* = B \langle i \rangle$, the vertices in $B \setminus (B^* \cup \varphi)$ form an interval and both (BS3) and (BS4) hold.

By Proposition 6 we are left with an interval $[j - |S| + 1, j]$ that satisfies (I2). Note that φ includes the center $j - |S|$ of the star B^{**} , but does not include j . Consequently, $B[j - |S|, j]$ consists of isolated vertices after the blue-star embedding, and j is not in edge-conflict with s . Hence, we can embed S explicitly onto $[j, j - |S| + 1]$. \square

Proposition 25. *If $B[j - |S| + 1, j]$ is a star, $2 \leq |S| = |B^{**}|$, and $\{i, j\} \notin E(B)$, then R and B admit an ordered plane packing onto $[i, j]$.*

Proof. By Lemma 17, we may assume that $B[i, j - |S|]$ is not a star. By Lemma 15 and Lemma 23, we may assume that $\deg_{R^-}(r) \geq 1$ and R^- is not a star. By Lemma 16, we may assume that $|S| \geq 2$. Due to LSFR, the center and the root of B^{**} are each located at either $j - |S| + 1$ or j , but B^{**} may be either a central-star or a dangling star. We distinguish two cases.

Case 1 S is not a central-star or B^{**} is a central-star. If necessary, flip B^{**} to put its center at j . We will later perform a blue-star embedding of S from j with $\varphi = (j - |S|, \dots, z)$.

Let us first check the conditions for the blue-star embedding. (BS1) follows from the condition that $\{i, j\} \notin E(B)$. For the other conditions, consider first the case that B^{**} is a central-star. If the parent p of j is in B then since B^{**} is maximal, p must have a subtree other than B^{**} . By LSFR and $\deg_S(s) \leq |S| - 1$, we have that $p < z$. Hence, regardless of whether p is in B , we know that $B \setminus (B^{**} \cup \varphi)$ forms an interval. (BS3) and (BS4) follow. For (BS2), on the one hand we have $|S| = |B^{**}| < |B^{**}| + \deg_S(s)$. On the other hand we have $|B^{**}| + 1 + \deg_S(s) \leq 2|S| \leq |I| - 1$.

Otherwise, B^{**} is a dangling star. Then (BS4) is satisfied (with $B^+ := B^{**}$) and (BS3) is satisfied by the assumption of Case 1 and the choice of φ . For (BS2), on the one hand we have $|S| = |B^{**}| \leq |B^{**}| - 1 + \deg_S(s)$ since $\deg_S(s) \geq 1$. On the other hand we have $|B^{**}| + \deg_S(s) < 2|S| \leq |I| - 1$.

Before performing this blue-star embedding, we modify the embedding of $B \langle i \rangle$. Since $B \langle i \rangle \neq B \langle j \rangle$ this does not affect the validity of the preconditions of the blue-star embedding. We want to ensure the following: if $[i, j - |S|]$ is in degree-conflict with R^- after the blue-star embedding, then $B \langle i \rangle$ is a star before the blue-star embedding. We proceed as follows.

The interval that the blue-star embedding will leave for R^- consists of $B[i, z - 1]$, followed by $\deg_S(s)$ isolated vertices (all in edge-conflict). Suppose that this interval would be in degree-conflict for embedding R^- . Then $B[i, x] := B[i, z - 1 \langle i \rangle]$ is a central-star. If $B[i, x] = B \langle i \rangle$ then we do nothing. Otherwise, $B[i, x]$ is rooted at i . Let p be the parent of i in B . By 1SR we have $B[i, p] = B \langle i \rangle$. If $\deg_B(p) = 1$ then $B \langle i \rangle$ is a dangling star and we do nothing. Otherwise, $\deg_B(p) \geq 2$. We claim that then $z \geq x + 2$. To prove the claim, suppose to the contrary that $z \leq x + 1$. Since $B \langle i \rangle \neq B \langle j \rangle$ and $|B \langle j \rangle| \geq |S|$ we know that $B \langle i \rangle \subseteq [i, j - |S|]$. By LSFR

and 1SR and $\deg_B(p) \geq 2$, p has at least one subtree B' besides $B[i, x]$ in $[i, j - |S|]$ with size $|B'| \geq |B[i, x]| = 1 + \deg_B(i)$. Since $z \leq x + 1$, we know that the blue-star embedding consumes p and all except at most one vertex of B' . Hence, $\deg_S(s) \geq 1 + |B'| - 1 \geq 1 + \deg_B(i)$. By the degree-conflict, we know that $\deg_{R^-}(r) + \deg_B(i) \geq |R^-|$. Since every subtree of r in R^- has size at least $|S| \geq 1 + \deg_S(s)$, we get

$$\begin{aligned} \deg_{R^-}(r) &\geq |R^-| - \deg_B(i) \\ &\geq (\deg_{R^-}(r))(1 + \deg_S(s)) - \deg_B(i) \\ &\geq (\deg_{R^-}(r))(2 + \deg_B(i)) - \deg_B(i) \\ &> 2 \deg_{R^-}(r), \end{aligned}$$

a contradiction. This proves our claim that $z \geq x + 2$. Now let G_1, \dots, G_k be the subtrees of p from left to right. Note that $G_1 = B[i, x]$. We select a parameter $t \in \{1, \dots, k\}$ as follows. If $z = p$ then let $t = k$. If z coincides with the root of a subtree of p , then let t be such that z coincides with the root of G_{t+1} . Otherwise, let t be such that z is contained in G_t . Then $t \geq 2$ since $z \geq x + 2$. Modify the embedding of $B\langle i \rangle$ as follows. Flip the embedding of each subtree G_t, \dots, G_k individually. Simultaneously shift each subtree G_t, \dots, G_k one position to the right and shift p to the position before G_t . In this modified embedding, $B[i, z - 1]$ satisfies LSFR and 1SR and $B[i, z - 1]\langle i \rangle$ is not a central-star, as intended.

Now perform the blue-star embedding of S with the parameters listed above. Recursively embed R^- onto $[i, j - |S|]$. This works unless there is a conflict. There can be no edge-conflict since $B\langle i \rangle \neq B\langle j \rangle$ and by (I1). If there is a degree-conflict, then $B[i, z - 1]\langle i \rangle$ is a central-star centered at c and $\deg_{R^-}(r) + \deg_{B[i, z - 1]}(c) \geq |R^-|$. By the modification of the embedding described above, we know that $B\langle i \rangle$ was a (possibly larger) star before the blue-star embedding. Undo the blue-star embedding. We have $\deg_{R^-}(r) + |B\langle i \rangle| - 1 \geq |R^-|$. We distinguish two subcases.

Case 1.1 $|B\langle i \rangle| + \deg_{R^-}(r) \leq |I| - 1$. Flip B^{**} to put its center at $j - |S| + 1$. If necessary, flip $B\langle i \rangle$ to put its center at i . First blue-star embed R^- from i with φ as the $\deg_{R^-}(r)$ leftmost vertices following $B\langle i \rangle$; and then embed S onto $[j, j - |S| + 1]$ using Algorithm 1. The conditions for the blue-star embedding are met: (BS1) follows from (I1) and (BS2) follows from $\deg_{R^-}(r) + |B\langle i \rangle| - 1 \geq |R^-|$ and the assumption of Case 1.1. (BS3) and (BS4) hold by choice of φ . The blue-star embedding replaces the center of B^{**} at $j - |S| + 1$ with an isolated vertex, but it does not affect j . Consequently, after the blue-star embedding $B[j - |S| + 1, j]$ consists of $|S| \geq 2$ isolated vertices, where j is not in edge-conflict with s . Thus, we can embed S onto $[j, j - |S| + 1]$.

Case 1.2 $|B\langle i \rangle| + \deg_{R^-}(r) \geq |I|$. By (I1), $B\langle i \rangle$ is not a central-star and must hence be a dangling star $B[i, y]$. Since $B[i, z - 1]\langle i \rangle$ is a central-star, $B\langle i \rangle$ is centered at i and $z \leq y$. Flip $B\langle i \rangle$ to place the center at y . Perform the original blue-star embedding of S from j again. Let us consider the interval $B[i, j - |S|]$ that remains for R^- . Since $z \leq y$, $B[i, j - |S|]$ is an independent set. At i we have the original root of $B\langle i \rangle$, which may be in edge-conflict with r . Each of the $\deg_S(s)$ rightmost vertices of $B[i, j - |S|]$ is in edge-conflict with r . Since $|B\langle j \rangle| \geq |B^{**}| \geq 2$ and $B\langle i \rangle \neq B\langle j \rangle$ we have $\deg_{R^-}(r) \geq |I| - |B\langle i \rangle| \geq 2$. Every subtree of r in R^- has size at least $|S|$, and hence we can embed one subtree explicitly on a prefix of $[i, j - |S|]$ (which takes care of the vertex i which is potentially in edge-conflict) and one subtree explicitly on a suffix of $[i, j - |S|]$ (which takes care of all $\deg_S(s)$ vertices which are in edge-conflict). The remaining vertices are not in edge-conflict, and so we can explicitly complete this partial embedding of R^- to a complete embedding of R^- .

Case 2 S is a central-star and B^{**} is a dangling star. Flip B^{**} if necessary to put its root at j . This preserves 1SR on B . We distinguish two cases.

Case 2.1 Every vertex in $B[i + 1, j - |S|]$ is a neighbor of j . Since $B\langle i \rangle \neq B\langle j \rangle$ the blue embedding

is completely determined. Flip the blue embedding at $[j - |S|, j]$. Embed s onto j and the children of s onto $[j - |S| + 1, j - 1]$. $B[i, j - |S|]$ consists of an isolated vertex at i (which is not in edge-conflict with r by (I3)) and a central-star $B[i + 1, j - |S|]$. Hence, we can embed R^- recursively onto $[i, j - |S|]$.

Case 2.2 Some vertex in $B[i + 1, j - |S| + 1]$ is not a neighbor of j . We first try the following. Use the red-star embedding to embed s to j and the children of s onto the rightmost $\deg_S(s)$ non-neighbors of j in $[i + 1, j]$. (RS1) holds due to $\{i, j\} \notin E(B)$. For (RS2) we have to show that there are at least $\deg_S(s)$ non-neighbors of j in $[i, j - 1]$. This is the case because B^{**} already contains $|B^{**}| - 2 = |S| - 2 = \deg_S(s) - 1$ non-neighbors of j , and the last vertex is provided by the assumption of this case. Embed R^- recursively onto $[i, j - |S|]$.

This works unless there is a conflict, in which case $B[i, j - |S|]\langle i \rangle$ is a central-star. As usual, this central-star cannot be in edge-conflict for r and is hence in degree-conflict. Since $B\langle i \rangle = B[i, j - |S|]\langle i \rangle$ both before and after the red-star embedding and since the red-star embedding either leaves $B\langle i \rangle$ untouched or replaces *only* its rightmost vertex by a vertex that is isolated in $B[i, j - |S|]$, we know that $B\langle i \rangle$ was a (dangling or central-)star before the blue-star embedding. Undo the red-star embedding. By the degree-conflict, we have $\deg_{R^-}(r) + |B\langle i \rangle| - 1 \geq |R^-|$. We proceed analogously to Case 1.1 and Case 1.2.

Case 2.2.1 $|B\langle i \rangle| + \deg_{R^-}(r) \leq |I| - 1$. Recall that the center of B^{**} is at $j - |S| + 1$. If necessary, flip $B\langle i \rangle$ to put its center at i . First blue-star embed R^- from i with φ as the $\deg_{R^-}(r)$ leftmost vertices following $B\langle i \rangle$; and then embed S onto $[j, j - |S| + 1]$ using Algorithm 1. The conditions for the blue-star embedding are met: (BS1) follows from (I1) and (BS2) follows from $\deg_{R^-}(r) + |B\langle i \rangle| - 1 \geq |R^-|$ and the assumption of Case 2.2.1. (BS3) and (BS4) hold by choice of φ and since R^- is not a star. The blue-star embedding replaces the center of B^{**} at $j - |S| + 1$ with an isolated vertex, but it does not affect j . Consequently, after the blue-star embedding $B[j - |S| + 1, j]$ consists of $|S| \geq 2$ isolated vertices, where j is not in edge-conflict with s . Thus, we can embed S onto $[j, j - |S| + 1]$.

Case 2.2.2 $|B\langle i \rangle| + \deg_{R^-}(r) \geq |I|$. By (I1), $B\langle i \rangle$ is not a central-star and must hence be a dangling star $B[i, y]$. Since the red-star embedding of S used only one vertex of $B[i, j - |S|]$ and since $B\langle i \rangle$ was a central-star after the red-star embedding, $B\langle i \rangle$ must be rooted at y and centered at i . Flip $B\langle i \rangle$ to place the center at y . Perform the original red-star embedding of S from j again. Let us consider the interval $B[i, j - |S|]$ that remains for R^- . $B[i, j - |S|]$ is an independent set. At i we have the original root of $B\langle i \rangle$, which may be in edge-conflict with r . The rightmost vertex of $B[i, j - |S|]$ is in edge-conflict with r . Since $|B\langle j \rangle| \geq |B^{**}| \geq 2$ and $B\langle i \rangle \neq B\langle j \rangle$ we have $\deg_{R^-}(r) \geq |I| - |B\langle i \rangle| \geq 2$. Every subtree of r in R^- has size at least $|S|$, and hence we can embed one subtree explicitly on a prefix of $[i, j - |S|]$ (which takes care of the vertex i which is potentially in edge-conflict) and one subtree explicitly on a suffix of $[i, j - |S|]$ (which takes care of the vertex $j - |S|$ which is in edge-conflict). The remaining vertices are not in edge-conflict, and so we can explicitly complete this partial embedding of R^- to a complete embedding of R^- . \square

Proposition 26. *If $B[j - |S| + 1, j]$ is a star, $2 \leq |S| = |B^{**}|$, $\{i, j\} \in E(B)$, and S is not a star, then R and B admit an ordered plane packing onto $[i, j]$.*

Proof. The presence of edge $\{i, j\} \in E(B)$ means that B is a tree, rooted at i or j . We distinguish two cases based on the root of B . By Lemma 23, we may assume that R^- is not a star.

Case 1 B is a tree rooted at i . We shall flip B , and show that $B[j - |S| + 1, j]$ is no longer a star after the flip. By LSFR, B^{**} is a smallest subtree of i . The largest subtree of i has size at least $|S| = |B^{**}|$, and so its root is outside of $[i, i + |S| - 1]$. Therefore, $B[i, i + |S| - 1]\langle i \rangle$ is an isolated vertex. Consequently, after flipping B , $B[j - |S| + 1, j]\langle j \rangle$ is an isolated vertex, and $B[j - |S| + 1, j]$ cannot be a star. If $B[i, j - |S|]$ is a star now, use Lemma 17 to find an ordered plane packing.

Otherwise, none of S , R^- , $B[i, j - |S|]$ and $B[j - |S| + 1, j]$ are stars, and we can use Lemma 10 to find an ordered plane packing.

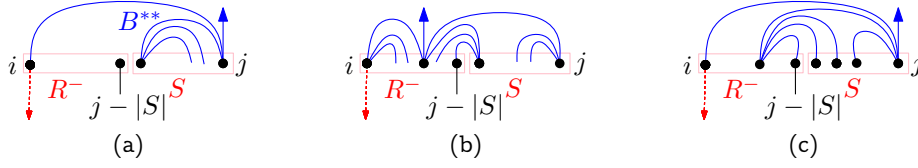


Figure 39: (a) $B^{**} = S$, $\{i, j\} \in E(B)$, and B is rooted at j . (b) When two subtrees of j are central-stars each with at least 2 vertices. (c) When j has a unique maximal subtree, and all other subtrees are singletons or not central-stars.

Case 2 B is a tree rooted at j . See Figure 39a. Since B is not a star, LSFR implies that B^{**} is a dangling star rooted at j . That is, $B[j - |S| + 1, j - 1]$ is a central-star, and by LSFR it is a largest subtree of j . Because S is a smallest subtree of r , we have $|S| \leq (|I| - 1)/2$, and so every subtree of j has size at most $|S| - 1 \leq (|I| - 3)/2$. Consequently, j has at least 3 subtrees in B . We distinguish subcases based on the subtrees of j .

Recall that j has a maximal subtree that is a central-star ($B[j - |S| + 1, j - 1]$). If j has another maximal subtree, then either this is a central-star (Case 2.2) or not (Case 2.1). Otherwise, $B[j - |S| + 1, j - 1]$ is the unique maximal subtree of j and either there exists another subtree of j that is a central-star on ≥ 2 vertices (Case 2.2) or every other subtree of j is a singleton or not a central-star (Case 2.3).

Case 2.1 j has two or more maximal subtrees, but not all of them are central-stars. Re-embed B using Algorithm 1 such that the subtree closest to j is *not* a central-star (we only change a tie-breaking rule in Algorithm 1). Then $B[j - |S| + 1, j]$ is no longer a star, and $B[i, j - |S|]$ does not become a star. Use Lemma 10 to find an ordered plane packing.

Case 2.2 Two or more subtrees of j are central-stars each with at least 2 vertices. Let $C_1 := B[j - |S| + 1, j - 1]$, which is central-star subtree of j with at least 2 vertices. Let C_2 be another subtree of j that is a central-star and has minimal size (possibly 1). We re-embed B as follows (see Figure 39b). Embed the root of B at $j - |C_1| - |C_2|$. Embed C_1 onto $[j, j - |C_1| + 1]$ and C_2 onto $[j - |C_1|, j - |C_1| - |C_2| + 1]$ each respecting 1SR. Embed all remaining subtrees onto $[i, j - |C_1| - |C_2| - 1]$ each respecting 1SR. Note that B does not obey 1SR because its root has subtrees on both sides. However, $B[i, j - |S|]$ and $B[j - |S| + 1, j]$ each satisfy both LSFR and 1SR. Furthermore, neither $B[i, j - |S|]$ nor $B[j - |S| + 1, j]$ is a star (since j has at least 3 subtrees); and $B[j - |S| + 1, j] \langle j - |S| + 1 \rangle$ is an isolated vertex.

Provisionally place r at i . We embed S recursively onto $[j - |S| + 1, j]$. There is no conflict for this embedding since $j - |S| + 1$ is isolated in $B[j - |S| + 1, j]$ and not adjacent to i . Embed R^- recursively onto $[i, j - |S|]$. This works because $B[i, j - |S|] \langle i \rangle$ is a singleton or not a central-star. Indeed, suppose to the contrary that $B[i, j - |S|] \langle i \rangle$ is a central-star. By construction, $B[i, j - |S|] \langle i \rangle = B[i, j - |C_1| - |C_2|]$ and contains the root of B . Hence, apart from C_1 and C_2 , all subtrees of the root of B are singletons. By the choice of C_2 , however, C_2 is also a singleton. Therefore the root of B has only one subtree with at least 2 vertices, contradicting our assumption.

Case 2.3 j has a unique maximal subtree, which is a central-star, and every other subtree is either a singleton or not a central-star. Recall that j has at least 3 subtrees. Re-embed B such that its root is at j , an arbitrary smallest subtree is embedded closest to j , and all other subtrees are embedded according to LSFR (all subtrees are embedded recursively by Algorithm 1). In particular, B^{**} is now the second subtree of j , counting from the right. See Figure 39c. As a result, $B[j - |S| + 1, j]$ is no longer a star, and $B[i, j - |S|]$ does not become a star. Note also that

$B[j - |S| + 1, j][j - |S| + 1]$ becomes an isolated vertex (it is a leaf of the dangling star B^{**}); and $B[i, j - |S|][i]$ is either an isolated vertex or not a central-star.

Provisionally place r at i . Embed S recursively onto $[j - |S| + 1, j]$. This works because $B[j - |S| + 1, j][j - |S| + 1]$ is locally isolated and not adjacent to i . Embed R^- recursively onto $[i, j - |S|]$. The recursive embedding of R^- works because $B[i, j - |S|][i]$ is either an isolated vertex (which is not adjacent to the blue vertex on which s was embedded) or not a central-star. \square

It remains to consider the case where $B[j - |S| + 1, j]$ is a star, $2 \leq |S| = |B^{**}|$, $\{i, j\} \in E(B)$, and S is a star. We deal with this case by handling the case where S is a star and $\{i, j\} \in E(B)$ in full generality.

Proposition 27. *If S is a star and $\{i, j\} \in E(B)$, then R and B admit an ordered plane packing onto $[i, j]$.*

Proof. Since $\{i, j\} \in E(B)$, B is a tree, rooted at i or j , and we can use symmetry by exchanging the roles of B and R (Figure 40a). Remove the embedding of B . Embed R using Algorithm 1, placing its root at j . Rename R to B and B to R . Define S to be a smallest subtree of R . Since B is rooted at j and B is not a star, there is no conflict for embedding R onto $[i, j]$.

Embedding R onto $[i, j]$ is handled by Lemma 10, Lemma 15, Lemma 16, Lemma 17, Lemma 23, or Proposition 24 unless the situation after the color exchange is as follows: $\deg_R(r) \geq 2$, $|S| \geq 2$, $B[i, j - |S|]$ is not a star, R^- is not a star, and (i) S is a star with $|S| \geq 2$ or (ii) $B[j - |S| + 1, j]$ is a star and the maximal star that contains $B[j - |S| + 1, j]$ has size exactly $|S|$. If S is not a star then (ii) holds and we can use Proposition 26 to find an ordered plane packing.

Otherwise, we are in Case (i) and S is a star. This means that the smallest subtree of both r and b is a star on at least two vertices and both R and B have at least two subtrees each. Denote by S_B a smallest subtree of B . By symmetry (possibly exchanging roles again), we may assume $|S_B| \geq |S|$.

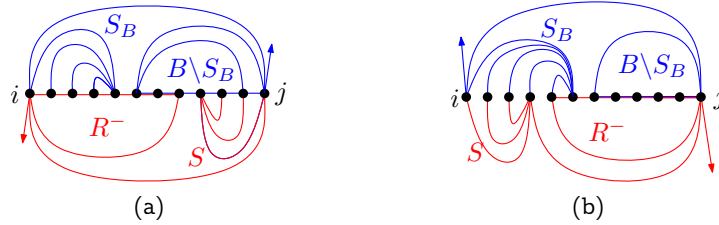


Figure 40: When R and B play symmetric roles.

We proceed as follows (Figure 40b). Re-embed B in the upper halfplane, placing b at i , S_B as the closest subtree, and the remaining subtrees according to LSFR.

We first explain how to embed S . We will do this in such a way that s is embedded on a vertex of S_B at $i + |S| - 1$. If S_B is a central-star, this re-embedding places its root and center at $i + |S_B|$. Since $|S_B| \geq |S|$, now $B[i, i + |S| - 1]$ is an independent set. Embed S explicitly onto $[i + |S| - 1, i]$. If S_B is a dangling star, the re-embedding places its root at $i + |S_B|$ and its center at $i + 1$. If $|S| = 2$, then embed s onto $i + 1$ and its child onto i . If $|S| \geq 3$ and S is a central-star, flip $B[i + 1, i + |S_B|]$ to put the root of S_B at $i + 1$ and the center at $i + |S_B|$ and embed s onto $i + |S| - 1$ and the children of s onto $[i - |S| - 2, i + 1]$. If $|S| \geq 3$ and S is a dangling star, flip $B[i + 1, i + |S_B| - 1]$ to put the center of S_B at $i + |S_B| - 1$ and embed s onto $i + |S| - 1$, its child s' onto i , and the children of s' onto $[i + 1, i + |S| - 2]$.

Next, embed R^- recursively onto $[j, j - |R^-| + 1]$. Since s was not embedded at i , the only obstacle for this recursive embedding is a possible conflict, in which case $B^* = B[i + |S|, j][j]$ is a central-star. Since $i + |S| - 1$ (which is where we embedded s) is adjacent only to vertices of S_B and possibly b , and since none of these vertices are part of B^* , the conflict must be a

degree-conflict. Then $|B^*| \geq 3$. As the root b of B is not in $[j, j - |R^-| + 1]$, we can reorder the subtrees of $B \setminus S_B$ arbitrarily without having to worry about LSFR on $[j, j - |R^-| + 1]$. Therefore, we may suppose that *all* subtrees of b are central-stars on ≥ 3 vertices and each of them leads to a degree-conflict when taking the role of $B^* = B\langle j \rangle$ above. Given that there are at least two such substars, we may as well choose a smallest one, S_B to have its center at j . Any other substar can take the role intended for S_B in Figure 40b originally, its leaves being paired up with S .

We claim that then there is no degree-conflict for embedding R^- onto $[j, j - |R^-| + 1]$ recursively. For such a degree-conflict to occur we need $\deg_{R^-}(r) + |S_B| - 1 \geq |R^-|$. So let us argue that this does not happen.

By the choice of S as a minimal size subtree of r , we have $\deg_R(r) \leq (|R| - 1)/|S|$. As S_B is a smallest of at least two subtrees of b , we have $|S_B| \leq (|B| - 1)/2 = (|R| - 1)/2$. Together this yields

$$\begin{aligned} \deg_{R^-}(r) + |S_B| &= \deg_R(r) - 1 + |S_B| \\ &\leq \frac{|R| - 1}{|S|} - 1 + \frac{|R| - 1}{2} \\ &= \frac{|R||S| + 2|R| - |S| - 2}{2|S|} - 1 \\ &= \frac{|R||S| + 2|R| - 3|S| - 2}{2|S|}. \end{aligned}$$

We want to show $\deg_{R^-}(r) + |S_B| \leq |R^-|$. So consider the expression

$$\begin{aligned} |R^-| - (\deg_{R^-}(r) + |S_B|) &= |R| - |S| - (\deg_{R^-}(r) + |S_B|) \\ &\geq |R| - |S| - \frac{|R||S| + 2|R| - 3|S| - 2}{2|S|} \\ &= \frac{|R||S| - 2|R| - 2|S|^2 + 3|S| + 2}{2|S|} \\ &= \frac{(|S| - 2)(|R| - 2|S| - 1)}{2|S|}, \end{aligned}$$

which is non-negative because $2 \leq |S| \leq (|R| - 1)/2$. This proves our claim and shows that there is no degree-conflict for embedding R^- onto $[j, j - |R^-| + 1]$ recursively. Therefore at least one of the two options provides an ordered plane packing as claimed. \square

Lemma 16, Proposition 24, Proposition 25, Proposition 26, and Proposition 27 together prove the following.

Lemma 28. *If $B[j - |S| + 1, j]$ is a star, then R and B admit an ordered plane packing onto $[i, j]$.*

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Next, we handle the case where S is a star. We may assume that $B[i, j - |S|]$ and $B[j - |S| + 1, j]$ are not stars. The graph R^- is also not a star and $|S| \geq 2$.

Proposition 29. *If S is a star and $\{i, j\} \notin E(B)$, then R and B admit an ordered plane packing onto $[i, j]$.*

Proof. We may assume $\deg_R(r) \geq 2$ by Lemma 15. S can be a central-star or a dangling star. We handle these cases separately. By Lemma 16, we may assume that $|S| \geq 2$. Let x be such that $|R^-| = |[i, x]|$. Flip $B\langle j \rangle$ if necessary to put the root at j . We use the following observation several times.

Observation 30. Suppose that we embedded s on a vertex of $B\langle j \rangle$ and that at most $|[i, x]| - 1$ rightmost vertices of $B[i, x]$ have been replaced by locally isolated vertices. Then $[i, x]$ is not in edge-conflict for embedding R^- onto $[i, x]$.

Proof. Suppose to the contrary that $[i, x]$ is in edge-conflict for embedding R^- . Let $y \leq x$ such that $B[i, y] = B[i, x]\langle i \rangle$. Then the root of $B[i, y]$ is in edge-conflict with r . It cannot be due to an edge to s since $B\langle i \rangle \neq B\langle j \rangle$. Hence, it must have an edge to the outside of $[i, j]$. By 1SR and LSFR, $B\langle i \rangle$ must then also be a (possibly larger) central-star whose root is in edge-conflict with r . This contradicts the peace invariant for embedding R onto I and thus concludes the proof. \square

Case 1 S is a central-star. Since $\{i, j\} \notin E(B)$ we have $B\langle i \rangle \neq B\langle j \rangle$ and hence this does not change the blue vertex at i . Use the red-star embedding to embed s onto j and the children of s onto the rightmost $\deg_S(s)$ non-neighbors of j in $[i + 1, j - 1]$. If $B[i, x]$ is a star now, then it was also a star before the red-star embedding (which may have modified $B[i, x]$), and we can find an ordered plane packing with Lemma 17. Otherwise, recursively embed R^- onto $[i, x]$. By the placement invariant and since $\{i, j\} \notin E(B)$, the placement invariant for the recursive embedding of R^- holds. Hence, the embedding of R^- fails only if (1) there is a conflict for embedding R^- onto $[i, x]$. For the embedding of S , (RS1) holds and so the embedding works unless (RS2) fails, i.e. unless (2) $\deg_S(s) + \deg_B(j) \geq |I| - 1$. We deal with (1)-(2) next.

Case 1.1 There is a conflict for embedding R^- onto $[i, x]$. Let $y \leq x$ such that $B[i, y] = B[i, x]\langle i \rangle$. Then $B[i, y]$ is a central-star rooted at a vertex b^* . By Observation 30, the conflict for embedding R^- onto $[i, x]$ is a degree-conflict. In other words, $\deg_{[i, x]}(b^*) + \deg_{R^-}(r) \geq |R^-|$. Consequently, $|B[i, y]| \geq |R^-| - \deg_{R^-}(r)$. Additionally, $\deg_{R^-}(r) \geq 2$ since $B[i, x]$ is not a star and $|B[i, y]| \geq 3$ by Lemma 2. Revert to the original blue embedding. See Figure 41a. Note that $B[i, y]$ is still a central-star. We distinguish two cases.

Case 1.1.1 $B\langle j \rangle$ is not a central-star. Then in particular $|B\langle j \rangle| \geq 3$. Since $\deg_R(r) \geq 2$ and $|B[i, y]| \geq |R^-| - \deg_{R^-}(r)$ we get by Lemma 9 that $|B[i, y]| \geq |S|$. If $B[i, y]$ is rooted at y then $B\langle i \rangle = B[i, y]$ by 1SR and we can flip $B\langle i \rangle$ to put the root (and center) at i . $B[i, i + |S| - 1]$ is now a small blue star. Flip $B\langle j \rangle$ to put its root at the left and embed R onto $[j, i]$ with Lemma 28. This works because j is not in edge-conflict with r and $B\langle j \rangle$ is not a central-star.

Case 1.1.2 $B\langle j \rangle$ is a central-star. Flip $B\langle j \rangle$ if necessary to put its root (and center) at j . If $|B\langle j \rangle| \geq |S|$ then use Lemma 28 to find an ordered plane packing. Otherwise $|S| \geq |B\langle j \rangle| + 1$. We distinguish two cases.

Case 1.1.2.1 $B\langle i \rangle$ is a central-star. Let z such that $B[i, z] = B\langle i \rangle$ and note that $z \geq y$. If necessary, flip $B[i, z]$ to put its root at i . By the peace invariant, i is not in edge-conflict with r . Since i is in degree-conflict with r for embedding R^- onto $[i, x]$ we have $\deg_B(i) + \deg_{R^-}(r) \geq |R^-|$.

In our first attempt at embedding R , we embedded S from j using a red-star embedding and tried to embed R^- onto $[i, x]$. Since $|S| \geq 2$, the red-star embedding moved all (possibly zero) children of j in $B\langle j \rangle$ to a suffix of $[i, x]$. Since there was a degree-conflict for the embedding of R^- onto $[i, x]$, it follows that $\deg_{R^-}(r) > |B\langle j \rangle| - 1$. Let h such that $B[h, j] = B\langle j \rangle$. See Figure 41b.

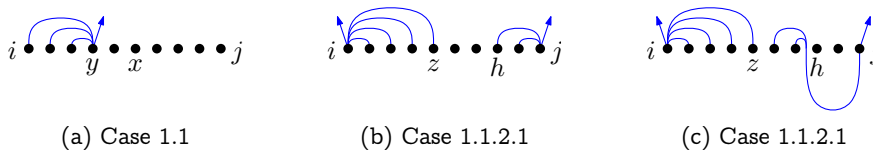


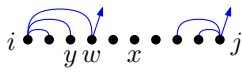
Figure 41: The case analysis in the proof of Proposition 29 (Part 1/3).

We know $\deg_B(i) + \deg_R(r) \leq |I| - 1$ by the peace invariant. It follows that $|B(i)| + \deg_{R^-}(r) \leq |I| - 1$. Combining this with the degree-conflict at i , we obtain $|R^-| \leq |B[i, y]| + \deg_{R^-}(r) \leq |B(i)| + \deg_{R^-}(r) \leq |I| - 1$. Hence, (BS2) is satisfied and we can perform a blue-star embedding to embed R^- onto $[i, j]$ (which will not embed any vertex onto j). Before doing so, modify the blue embedding by simultaneously shifting $B[h, j-1]$ to $[z+1, z+j-h]$ (redrawing the edges to j with biarcs) and $B[z+1, h-1]$ to $[z+j-h+1, j-1]$. See Figure 41c. Since $\deg_{R^-}(r) > |B(j)| - 1$, the blue-star embedding will embed a vertex on every child of $B(j)$. Complete the embedding by placing s at j and the children of s onto the remainder.

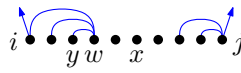
Case 1.1.2.2 $B(i)$ is not a central-star. Let w such that $B(i) = B[i, w]$. Since $B[i, y]$ is a central-star, by 1SR $B[i, y]$ must be rooted at i and $B[i, w]$ must be rooted at w . See Figure 42a.

We claim that $\deg_B(w) = 1$. Towards a contradiction, suppose that $\deg_B(w) \geq 2$. Recall that $\deg_B(i) + \deg_{R^-}(r) \geq |R^-|$. Since S is a smallest subtree of r in R , we have $\deg_{R^-}(r) \leq (|R^-| - 1)/|S| \leq |R^-|/|S|$. Hence, $\deg_B(i) \geq |R^-| - \deg_{R^-}(r) \geq (1 - 1/|S|)|R^-|$. By LSFR, $|B(w)| \geq 1 + 2(1 + \deg_B(i)) = 3 + 2\deg_B(i)$ and hence $|B(w)| \geq 3 + (2 - 2/|S|)|R^-|$. Since $|R^-| + |S| = |I| \geq |B(w)|$ and $|R^-| \geq |S|$, we obtain $|S| \geq 3 + (2 - 2/|S|)|R^-| - |R^-| = 3 + (1 - 2/|S|)|R^-| \geq 3 + |S| - 2 = |S| + 1$, a contradiction. The claim follows.

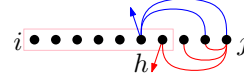
Since $B[i, y]$ is a central-star rooted at i , by LSFR $B(i) = B[i, w]$ is a star centered at i and rooted at w . If $w \geq x$ then we can use Lemma 17 to find an ordered plane packing. Otherwise, $w \leq x - 1$.



(a) Case 1.1.2.2



(b) Case 1.1.2.2



(c) Case 1.2

Figure 42: The case analysis in the proof of Proposition 29 (Part 2/3).

Since there was a conflict for the original embedding, the red-star embedding of S from j embeds a child of s onto all blue vertices originally at $[w, x]$. Flip $B[i, w]$ to put its root at i and center at w . See Figure 42b. Execute the red-star embedding of S from j again. This embeds a child of s onto the center of $B(i)$ at w and hence the remaining vertices of $B(i)$ form an independent set. Consider the now-modified blue embedding at $[i, x]$. The leftmost vertex of $B[i, x]$ is the original root of $B(i)$ and may be in edge-conflict with r . The suffix of $[i, x]$ of size $\deg_B(j) \leq |S| - 2$ is formed by blue vertices adjacent to j (which is where we embedded s) that were placed there by the red-star embedding of S from j . All of these blue vertices are in edge-conflict with r . However, by the original degree-conflict, we know that $\deg_{R^-}(r) \geq 2$ and hence we can find an explicit embedding of R^- onto $[i, x]$ that avoids placing the root at i or at the suffix of size $|S| - 2$. This uses that all subtrees of r in R^- have size at least $|S|$.

Case 1.2 $\deg_S(s) + \deg_B(j) \geq |I| - 1$. Then $\deg_B(j) \geq |I| - 1 - \deg_S(s) = |I| - |S| = |R^-| \geq (|I| + 1)/2$ and hence $B(j)$ has a leaf. Let h such that $B[h, j] = B(j)$. Then $|B[h, j]| > |S|$ and so $h \leq x$. If $B[h, j]$ is a star, then we flip $B[h, j]$ if necessary to put its center at j and use Lemma 28 to find an ordered plane packing. Otherwise, $B[h, j]$ is not a star. We claim that then $h < x$. Indeed, if $h = x$, then $\deg_B(j) \leq |B[x, j]| - 2 = |S| + 1 - 2 = |S| - 1$ and so $\deg_S(s) \geq |I| - 1 - \deg_B(j) \geq |I| - 1 - |S| + 1 = |R^-|$, a contradiction. The claim follows. Flip $B[h, j]$ to put the root on the left. This places a leaf at j . Embed s onto j and the children of s onto $[j-1, x+1]$. Embed R^- recursively onto $[x, i]$. See Figure 42c. The placement invariant holds since $h < x$ and h is the only vertex incident to j (which is where we embedded s). By LSFR and since $B[h, j]$ is not a star, $B[i, x] \setminus \{x\}$ is not a central-star. Hence the peace invariant holds and we can complete the packing.

Case 2 S is a dangling star. Then it is rooted at the child q of s . Let $Q = t(q)$. We will embed R similarly to Case 1. Let h such that $B[h, j] = B\langle j \rangle$. We distinguish two cases.

Case 2.1 Suppose that $B[h, j]$ is not a central-star. Then in particular $|B[h, j]| \geq 3$. Let h' be the rightmost neighbor of j in $[i, j-1]$. If $h' \leq x$, then embed s onto $j - |S| + 1$, q onto j , and the children of q onto $[j-1, j - |S| + 2]$. See Figure 43a. Otherwise, embed s onto $h' + 1$, q onto j , and embed a child of q onto every blue vertex of $[h' + 2, j-1]$. Use the red-star embedding to embed the remaining vertices onto the rightmost $\deg_Q(q) - |[h' + 2, j-1]|$ non-neighbors of j of $[i+1, h']$. In either case, embed R^- recursively onto $[i, x]$. The embedding of R^- works unless (1) there is a conflict for embedding R^- onto $[i, x]$. The embedding of S works unless (RS2) fails, i.e. unless (2) $\deg_Q(q) + \deg_B(j) \geq |I| - 2$.

Case 2.1.1 There is a conflict for embedding R^- onto $[i, x]$. Let $y \leq x$ such that $B[i, y] = B[i, x]\langle i \rangle$. Then $B[i, y]$ is a central-star. By Observation 30, the conflict for embedding R^- onto $[i, x]$ is a degree-conflict, and hence $|B[i, y]| \geq |R^-| - \deg_{R^-}(r)$. Following the reasoning in Case 1.1.1, we see that $|B[i, y]| \geq |S|$ and hence $B[i, i + |S| - 1]$ is a small blue star after flipping $B[i, y]$ if necessary. Flip $B[h, j]$ to put its root at h and use Lemma 28 to embed R onto $[j, i]$. This works because j is not in edge-conflict with r and $B\langle j \rangle$ is not a central-star.

Case 2.1.2 $\deg_Q(q) + \deg_B(j) \geq |I| - 2$. This case is similar to Case 1.2. Since $|S| \leq (|I| - 1)/2$ we have $\deg_Q(q) = |S| - 2 \leq (|I| - 5)/2$. Then $\deg_B(j) \geq |I| - 2 - \deg_Q(q) \geq |I| - 2 - (|I| - 5)/2 = (|I| + 1)/2$. Since $B[h, j]$ is not a central-star, we get $h < x$ as in Case 1.2. Let λ be the number of leaf children of j . Then $1 + \lambda + 2((\deg_B(j) - \lambda)) \leq |B\langle j \rangle| \leq |I| - 1$. Since $\deg_B(j) \geq (|I| + 1)/2$ it follows that $1 - \lambda + |I| + 1 \leq |I| - 1$ and hence $\lambda \geq 3$. Flip $B[h, j]$ to put its root at h . Since h now has λ leaf children in $B[h, j]$, in particular $j-1$ and j are leaves. Embed s onto j , q onto $j-1$, and the children of q onto $[j-2, x+1]$. Embed R^- recursively onto $[x, i]$. Since $B[h, j]$ is not a star by assumption and by LSFR, $B[i, x]\langle x \rangle$ is not a central-star on at least two vertices. Hence the peace invariant holds.

Case 2.2 Suppose that $B[h, j]$ is a central-star. If $h \leq x+1$ then $B[x+1, j]$ is a star and we can find an ordered plane packing by Lemma 28. Otherwise $h \geq x+2$. Flip $B\langle h-1 \rangle$ if necessary to put its root at $h-1$. Embed s onto j , q onto $h-1$, and a child of q on every vertex in $[h, j-1]$. Use the red-star embedding to embed the remaining children of q onto the rightmost $\deg_Q(q) - |[h, j-1]|$ non-neighbors of $h-1$ in $[i+1, h-2]$. Embed R^- recursively onto $[i, x]$. See Figure 43b for the situation before the cleanup step of the red-star embedding. The embedding of R^- works unless (1) there is a conflict for embedding R^- onto $[i, x]$. The embedding of S works unless (RS2) fails, i.e. unless (2) $\deg_Q(q) + \deg_B(h-1) \geq |I| - 2$.

Case 2.2.1 There is a conflict for embedding R^- onto $[i, x]$. Let $y \leq x$ such that $B[i, y] = B[i, x]\langle i \rangle$. Then $B[i, y]$ is a central-star. By Observation 30, the conflict is a degree-conflict. Revert to the original blue embedding (before the red-star embedding in Case 2.2) and note that $B[i, y]$ is still a central-star. We proceed similarly to Case 1.1.2.

Case 2.2.1.1 $B\langle i \rangle$ is a central-star. Let z such that $B[i, z] = B\langle i \rangle$ and note that $z \geq y$. If necessary, flip $B[i, z]$ to put its root at i . By the peace invariant, i is not in edge-conflict with r . Since i

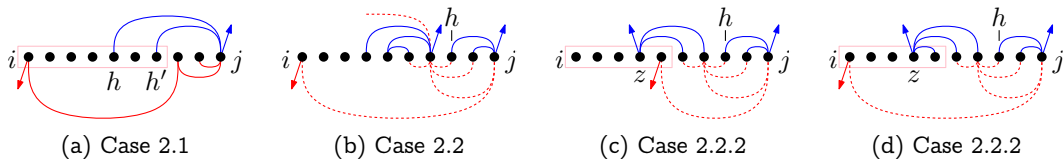


Figure 43: The case analysis in the proof of Proposition 29 (Part 3/3).

is in degree-conflict with r for embedding R^- onto $[i, x]$ we have $\deg_B(i) + \deg_{R^-}(r) \geq |R^-|$.

We blue-star embed R^- starting from i with $\varphi = (z+1, \dots)$. Let us argue that the conditions for the blue-star embedding hold. The peace invariant guarantees (BS1) and $\deg_B(i) + \deg_{R^-}(r) \leq |I| - 1$. It follows that $|B(i)| + \deg_{R^-}(r) \leq |I| - 1$, which is the second inequality of (BS2). The first inequality of (BS2) holds by the degree-conflict condition. (BS3) holds by construction, making (BS4) trivial. Hence, the conditions are satisfied and we can perform the blue-star embedding as described.

Since we attain the first inequality in (BS2) strictly, the blue-star embedding does not exhaust all vertices in $B[i, z]$. Indeed, $\deg_B(i) \geq |R^-| - \deg_{R^-}(r)$, while the blue-star embedding embeds only $|R^-| - \deg_{R^-}(r) - 1$ vertices on the neighbors of i . Perform the blue-star embedding of R^- onto $[i, j]$. This leaves an interval containing j (since the blue-star-embedding always leaves at least one vertex) and at least one locally isolated vertex (originating from $B[i+1, z]$). Embed s onto j , q onto this locally isolated vertex, and the children of q onto the remainder to complete the embedding.

Case 2.2.1.2 $B(i)$ is not a central-star. We proceed similarly to Case 1.1.2.2. Let w such that $B(i) = B[i, w]$. The exact same argument as in Case 1.1.2.2 shows that $B[i, w]$ is a star rooted at w and centered at i . If $w \geq x$ then we can use Lemma 17 to find an ordered plane packing. Otherwise, $w \leq x - 1$.

Since there was a conflict for the original embedding, the red-star embedding of (the remainder of) Q from $h-1$ embeds a child of q onto all blue vertices originally at $[w, x]$. Flip $B[i, w]$ to put its root at i and center at w . Embed s onto j , q onto $h-1$, and a child of q onto all vertices in $[h, j-1]$. Execute the red-star embedding of the remainder of Q from $h-1$ onto $[h-2, i+1]$ again. This embeds a child of s onto the center of $B(i)$ and hence the remaining vertices form an independent set. Consider the now-modified blue embedding at $[i, x]$. The leftmost vertex of $B[i, x]$ is the original root of $B(i)$ and may be in edge-conflict with r . We embedded a child of q onto all neighbors of j (which is where we embedded s), and hence there are no further edge-conflicts. Hence, we can embed R^- explicitly onto $[x, i]$.

Case 2.2.2 $\deg_Q(q) + \deg_B(h-1) \geq |I| - 2$. Let z such that $B[z, h-1] = B(h-1)$. It is possible that $z = i$ and $B(i) = B(h-1)$. Analogously to Case 2.1.2 we get $\deg_B(h-1) \geq (|I| + 1)/2$ and that $h-1$ has at least 3 leaf children. It follows that $z < x$. Recall that $h \geq x + 2$. Flip $B[z, h-1]$ to put its root at z . If $z = i$ and $B[i, x]$ is now a star, use Lemma 17 to find an ordered plane packing. Otherwise, flipping $B[z, h-1]$ placed a leaf child of z at $h-1$. Embed s onto j , q onto $h-1$, and the children of q onto $[j-1, h]$ and $[h-2, x+1]$. This works because $z < x$ and $h \geq x + 2$.

We first try to embed R^- recursively onto $[x, i]$. See Figure 43c. Since $z < x$, this works unless $B[i, x](x)$ is a central-star, which implies that $B[z, h-1]$ is a central-star by LSFR. In this scenario we already handled the case $z = i$ and so we may assume $B(i) \neq B(z)$. Embed R^- recursively onto $[i, x]$. See Figure 43d. By the placement invariant, this works unless there is a conflict for embedding R^- onto $[i, x]$.

So suppose there is a conflict for embedding R^- onto $[i, x]$. Since $z < x$ and $B(i) \neq B(z)$, we have $B[i, x](i) = B(i)$ and hence $B(i)$ is a central-star. By the peace invariant, the root of $B(i)$ is not in edge-conflict with r . Flip $B(i)$ if necessary to put its root at i . Then i is in degree-conflict with r and hence $\deg_B(i) + \deg_{R^-}(r) \geq |R^-|$. Adding this inequality to the inequality in the assumption (replacing $h-1$ by z due to our flipping), we get $\deg_B(i) + \deg_{R^-}(r) + \deg_Q(q) + \deg_B(z) \geq |I| - 2 + |R^-|$. Since $B(i)$, $B(z)$, and $B(j)$ are all different we have $\deg_B(i) + \deg_B(z) \leq |I| - 3$. Hence $\deg_{R^-}(r) + \deg_Q(q) \geq |I| - 2 + |R^-| - |I| + 3 = |R^-| + 1$. Since $|S| = \deg_Q(q) + 2$ we get $|S| + \deg_{R^-}(r) \geq |R^-| + 3$. Since S is a smallest subtree of r , we have $\deg_{R^-}(r) \leq (|R^-| - 1)/|S|$ and hence $|R^-| \geq |S| \deg_{R^-}(r)$. It follows that $|S| + \deg_{R^-}(r) \geq |S| \deg_{R^-}(r) + 3$, which has no solution for $|S| \geq 1$ and $\deg_{R^-}(r) \geq 1$. We conclude that there is no conflict for embedding R^- onto $[i, x]$, as desired. \square

Propositions 29 and 27 together prove the following.

Lemma 31. *If S is a star, then R and B admit an ordered plane packing onto $[i, j]$.*

Finally, Lemmata 10, 17, 23, 28, and 31 together prove Theorem 3.

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